Quantum cohomology of minuscule homogeneous spaces

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Abstract

We study the quantum cohomology of (co)minuscule homogeneous varieties under a unified perspective. We show that three points Gromov-Witten invariants can always be interpreted as classical intersection numbers on auxiliary homogeneous varieties. Our main combinatorial tools are certain quivers, in terms of which we obtain a quantum Chevalley formula and a higher quantum Poincaré duality. In particular we compute the quantum cohomology of the two exceptional minuscule homogeneous varieties.

1 Introduction

The quantum cohomology of complex homogeneous spaces has been studied by many people since the fundamental works of Witten and Kontsevich-Manin – see [Fu] for a survey. The (small) quantum cohomology ring of Grassmannians was investigated by Bertram with the help of Grothendieck's Quot schemes [Be], a method which can be applied in other situations but can be quite technical. An important breakthrough, which greatly simplified the computations of the quantum products in many cases, was the observation by Buch that the Gromov-Witten invariants of certain homogeneous spaces are controlled by classical intersection numbers, but on certain auxiliary homogeneous varieties [Bu].

In this paper we make use of this idea in the general context of minuscule and cominuscule homogeneous varieties (see the next section for the definitions). We give a unified treatment of the quantum cohomology ring of varieties including ordinary and Lagrangian Grassmannians, spinor varieties, quadrics and also the two exceptional Hermitian symmetric spaces – the Cayley plane E_6/P_1 and the Freudenthal variety E_7/P_7 (in all the paper we use the notations of [Bou] for root systems). One of the conclusions of our study is the following presentation of the quantum cohomology algebras.

Theorem 1.1 *Let $X = G/P$ be a minuscule homogeneous variety. There exists a minimal homogeneous presentation of its integer cohomology of the form*

$$H^*(X) = \mathbb{Z}[H, I_{p_1+1}, \dots, I_{p_n+1}]/(R_{q_1+1}, \dots, R_{q_r+1}),$$

and one can choose the relation of maximal degree R_{q_r+1} so that the quantum cohomology ring of X is

$$QH^*(X) = \mathbb{Z}[q, H, I_{p_1+1}, \dots, I_{p_n+1}]/(R_{q_1+1}, \dots, R_{q_r+1} + q).$$

Note that $q_r = h$ is the Coxeter number of G.

In fact this was already known for the classical minuscule homogeneous varieties (see [FP, KT1, KT2, ST]), so our contribution to this statement only concerns the exceptional cases, which are treated in section 5 (see Theorems 5.1 and 5.4). It is easy to deduce from these theorems the quantum product of any two Schubert classes in the two exceptional Hermitian symmetric spaces. This relies on the computations of the classical cohomology rings done in [IM] for the Cayley plane, and very recently in [NS] for the Freudenthal variety.

Our treatment of these homogeneous spaces relies on combinatorial tools that we develop in the general context of (co)minuscule varieties. Namely, we use the combinatorics of certain quivers, first introduced in

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[Pe2], to give a convenient interpretation of Poincaré duality on a (co)minuscule $X = G/P$. We deduce a nice combinatorial version of the quantum Chevalley formula (Proposition 4.1).

As we mentionned, this relies on the interpretation of degree one Gromov-Witten invariants as classical intersection numbers on the Fano variety of lines on X, which remains G-homogeneous. Generalizing the case by case analysis of [BKT] for classical groups, we extend this interpretation to degree d invariants in Corollary 3.28: Gromov-Witten invariants of degree d are classical intersection numbers on certain auxiliary G-homogeneous varieties F_d.

In fact this statement gives a very special role to a certain Schubert subvariety Y_d of X. In particular, we are able to define, in terms of this variety, a duality on a certain family of Schubert classes. In degree zero this is just the usual Poincaré duality, which we thus extend to a "higher quantum Poincaré duality", see Proposition 4.7. We also obtain a combinatorial characterization, again in terms of quivers, of the minimal power of q that appears in the quantum product of two Schubert classes: see Corollary 4.12. This extends a result of Buch [Bu], which itself relied on a general study of this question by Fulton and Woodward [FW].

Contents

2 Minuscule and cominuscule varieties

We begin by reminding what are the (co)minuscule homogeneous varieties, and how their Chow ring can be computed, in particular in the two exceptional cases. Then we introduce quivers in relation with Poincaré duality. These tools will be useful for a unified approach of quantum cohomology.

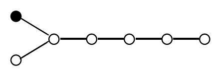
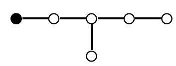
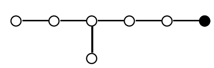
2.1 First properties

Let G be a simple complex algebraic group, and B a Borel subgroup. Let V be an irreducible representation of G, with highest weight ω. Recall that ω is *minuscule* if $|\langle\omega,\check{\alpha}\rangle| \leqslant 1$ for any root α (see [Bou], VI.1 exercice 24). This implies that ω is a fundamental weight, and all the weights of V are in the orbit of

ω under the Weyl group W. Correspondingly, the closed G-orbit $G/P \subset \mathbb{P}V$ is a *minuscule homogeneous variety.* Here and in the sequel P is a maximal parabolic subgroup of G.

A related notion is the following: a fundamental weight ω is *cominuscule* if $\langle \omega, \check{\alpha}_0 \rangle = 1$, where α_0 denotes the highest root. The two definitions characterize the same fundamental weights if G is simply laced. Equivalently, the *cominuscule homogeneous variety* $G/P \subset \mathbb{P}V$ is a G-Hermitian symmetric space in its minimal homogeneous embedding. (Beware that these are called *minuscule homogeneous varieties* in [LM].)

The list of cominuscule homogeneous varieties is the following. There are infinite series of classical homogeneous spaces plus two exceptional cases.

type	*variety*	*diagram*	*dimension*	*index*
A_{n-1}	$\mathbb{G}(k,n)$		$k(n-k)$	n
C_n	$\mathbb{G}_\omega(n,2n)$		$\frac{n(n+1)}{2}$	$n+1$
D_n	$\mathbb{G}_Q(n,2n)$		$\frac{n(n-1)}{2}$	$2n-2$
B_n, D_n	$\mathbb{Q}^m$		m	m
E_6	$\mathbb{OP}^2$		16	12
E_7	E_7/P_7		27	18

Recall that a simply-connected simple complex algebraic group G is determined by its Dynkin diagram, and that the conjugacy classes of its maximal parabolic subgroups P correspond to the nodes of this diagram. In the above array, the marked diagrams therefore represent such conjugacy classes, or equivalently the homogeneous varieties $X = G/P$. These are projective varieties, described in the second column, where the notation $\mathbb{G}(k,n)$ (resp. $\mathbb{G}_\omega(n,2n), \mathbb{G}_Q(n,2n)$) stands for the Grassmannian of k-planes in a fixed n-dimensional space (resp. the Grassmannian of maximal isotropic subspaces in a fixed symplectic or quadratic vector space of dimension $2n$). To be more precise, in the orthogonal case $\mathbb{G}_Q(n,2n)$ will be only one of the two (isomorphic) connected component of this Grassmannian; its minimal embedding is in the projectivization of a half-spin representation, and for this reason we sometimes call it a spinor variety; another consequence is that its index is twice as large as what one could expect. An m-dimensional quadric is denoted $\mathbb{Q}^m$ and, finally, the Cayley plane $\mathbb{OP}^2 = E_6/P_1$ and the Freudenthal variety E_7/P_7 are varieties described in subsections 2.3.1 and 2.3.2.

Beware that a quadric $\mathbb{Q}^m$ is minuscule if and only if its dimension m is even. Moreover there are two series of G-homogeneous varieties which are cominuscule but not minuscule, but actually their automorphism groups H are bigger than G and they are minuscule when considered as H-varieties: namely, the projective spaces of odd dimensions acted upon by the symplectic groups, and the maximal orthogonal Grassmannians for orthogonal groups in odd dimensions.

For a maximal parabolic subgroup P of G, the Picard group of G/P is free of rank one : $Pic(G/P) = \mathbb{Z}H$, where the very ample generator H defines the minimal homogeneous embedding $G/P \subset \mathbb{P}V$.

The index $c_1(G/P)$ is defined by the relation $-K_{G/P} = c_1(G/P)H$. This integer is of special importance with respect to quantum cohomology since it defines the degree of the quantum parameter q. A combinatorial recipe that allows to compute the index of any rational homogeneous space can be found in [Sn]. Geometrically, the index is related with the dimension of the Fano variety F of projective lines on $G/P \subset \mathbb{P}V$ [FP, p.50]:

$$\dim(F) = \dim(G/P) + c_1(G/P) - 3.$$

Equivalently, the Fano variety F_o of lines through the base point $o \in G/P$ has dimension

$$\dim(F_o) = c_1(G/P) - 2.$$

It was proved in [LM, Theorem 4.8] that when P is defined by a long simple root, F_o is homogeneous under the semi-simple part S of P. Moreover, the weighted Dynkin diagram of F_o can be obtained from that

of G/P by suppressing the marked node and marking the nodes that were connected to it. For example, if $G/P = E_7/P_7$ then F_o is a copy of the Cayley plane $E_6/P_6 \simeq E_6/P_1$, whose dimension is 16 – hence $c_1(E_7/P_7) = 18$.

We also observe that when G/P is minuscule, the relation

$$c_1(G/P) = e_{max}(G) + 1 = h$$

does hold, where $e_{max}(G)$ denotes the maximal exponent of the Weyl group of G and h is the Coxeter number [Bou, V, 6, definition 2].

2.2 Chow rings

In this section we recall some fundamental facts about the Chow ring of a complex rational homogeneous space $X = G/P$. This graded Chow ring with coefficients in a ring k will be denoted $A^*(G/P)_k$, and we simply write $A^*(G/P)$ for $A^*(G/P)_{\mathbb{Z}}$ (which coincides with the usual cohomology ring).

First, we recall the *Borel presentation* of this Chow ring with rational coefficients. Let W (resp. W_P) be the Weyl group of G (resp. of P). Let $\mathcal{P}$ denote the weight lattice of G. The Weyl group W acts on $\mathcal{P}$. We have

$$A^*(G/P)_{\mathbb{Q}} \simeq \mathbb{Q}[\mathcal{P}]^{W_P}/\mathbb{Q}[\mathcal{P}]^W_+,$$

where $\mathbb{Q}[\mathcal{P}]^{W_P}$ denotes the ring of W_P-invariants polynomials on the weight lattice, and $\mathbb{Q}[\mathcal{P}]^W_+$ is the ideal of $\mathbb{Q}[\mathcal{P}]^{W_P}$ generated by W-invariants without constant term (see [Bor], Proposition 27.3 or [BGG], Theorem 5.5).

Recall that the full invariant algebra $\mathbb{Q}[\mathcal{P}]^W$ is a polynomial algebra $\mathbb{Q}[F_{e_1+1}, \dots, F_{e_{max}+1}]$, where $e_1, \dots, e_{max}$ is the set $E(G)$ of exponents of G. If $d_1, \dots, d_{max}$ denote the exponents of S, we get that $\mathbb{Q}[\mathcal{P}]^{W_P} = \mathbb{Q}[H, I_{d_1+1}, \dots, I_{d_{max}+1}]$, where H represents the fundamental weight ω_P defining P; we denote it this way since geometrically, it corresponds to the hyperplane class. Now each W-invariant F_{e_i+1} must be interpreted as a polynomial relation between the W_P-invariants $H, I_{d_1+1}, \dots, I_{d_{max}+1}$. In particular, if e_i is also an exponent of the semi-simple part S of P, this relation allows to eliminate I_{e_i+1}. We thus get the presentation, by generators and relations,

$$A^*(G/P)_{\mathbb{Q}} \simeq \mathbb{Q}[H, I_{p_1+1}, \dots, I_{p_n+1}]/(R_{q_1+1}, \dots, R_{q_r+1}), \tag{1}$$

where $\{p_1, \dots, p_n\} = E(S) - E(G)$ and $\{q_1, \dots, q_r\} = E(G) - E(S)$. Note that $q_r = e_{max}$.

Over the integers, the Chow ring is a free $\mathbb{Z}$-module admitting for basis the classes of the Schubert varieties, the B-orbit closures in X. The Schubert subvarieties of X are parametrised by the quotient W/W_P. Recall that in any class, there exists a unique element of minimal length. We denote this set of representatives of W/W_P by W_X. For $w \in W_X$, let $X(w)$ be the corresponding Schubert subvariety of X. In particular, we let w_X the unique element of maximal length in W_X, such that $X(w_X) = X$. When P is maximal there is also a unique element of length $\ell(w_X) - 1$, defining the hyperplane class H with respect to the minimal homogeneous embedding of G/P. All the inclusion relations are given by the restriction to W_X of the Bruhat order on W. The *Hasse diagram* of G/P is the graph whose vertices are the Schubert classes, and whose edges encode the inclusion relations in codimension one.

To compute the product of two Schubert classes in the Schubert basis, we have the following tools:

1. *Poincaré duality* is known to define an involution of W_X, given by $w \mapsto w_0 w w_0 w_X$ (see e.g. [Kö]).

2. *the Chevalley formula* allows to multiply any Schubert class by the hyperplane class:

$$[X(w)] \cdot H = \sum_{w \to v = w s_\alpha} \langle \omega_P, \check{\alpha} \rangle [X(v)],$$

   where $w \to v$ must be an arrow in the Hasse diagram [Hi, V, coro 3.2].

Note that in the minuscule cases the integers $\langle \omega_P, \check{\alpha} \rangle$ are always equal to one. In particular the degree of each Schubert variety can be computed as a number of paths in the Hasse diagram.

2.3 The exceptional minuscule varieties

As we have seen, there exists two exceptional minuscule homogeneous spaces, the *Cayley plane* $E_6/P_1 \simeq E_6/P_6$, and the *Freudenthal variety* E_7/P_7. We briefly recall how they can be constructed and a few properties that will be useful in the sequel.

2.3.1 The Cayley plane

Let $\mathbf{O}$ denote the normed algebra of (real) octonions, and let $\mathbb{O}$ be its complexification. The space

$$\mathcal{J}_3(\mathbb{O}) = \left\{ \begin{pmatrix} c_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & c_2 & x_1 \\ x_2 & \overline{x}_1 & c_3 \end{pmatrix} : c_i \in \mathbf{C}, x_i \in \mathbb{O} \right\} \cong \mathbf{C}^{27}$$

of $\mathbb{O}$-Hermitian matrices of order 3, is the exceptional simple complex Jordan algebra.

The subgroup of $GL(\mathcal{J}_3(\mathbb{O}))$ consisting of automorphisms preserving a certain cubic form called determinant is the simply-connected group of type E_6. The Jordan algebra $\mathcal{J}_3(\mathbb{O})$ and its dual are the minimal representations of this group.

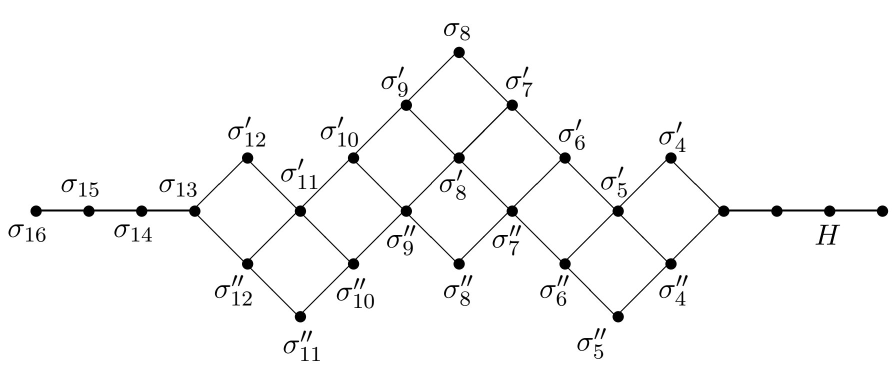
The Cayley plane can be defined as the closed E_6-orbit in $\mathbb{P}\mathcal{J}_3(\mathbb{O})$. It is usually denoted $\mathbb{OP}^2$ and indeed it can be interpreted as a projective plane over the octonions. In particular it is covered by a family of eight dimensional quadrics, which are interpreted as $\mathbb{O}$-lines.

In fact there exists only two other E_6-orbits in $\mathbb{P}\mathcal{J}_3(\mathbb{O})$. The open orbit is the complement of the determinantal cubic hypersurface Det. The remaining one is the complement of $\mathbb{OP}^2$ in Det, which can be interpreted as the projectivization of the set of rank two matrices in $\mathcal{J}_3(\mathbb{O})$. Observe that geometrically, Det is just the secant variety of the Cayley plane. For a general point $z \in Det$, the closure of the union of the secant lines to $\mathbb{OP}^2$ passing through z is a linear space Σ_z, and $Q_z := \Sigma_z \cap X$ is an $\mathbb{O}$-line (see [LV, Za] for more on this). We deduce the following statement:

Lemma 2.1 *Let q be a conic on $\mathbb{OP}^2$, whose supporting plane is not contained in $\mathbb{OP}^2$. Then q is contained in a unique $\mathbb{O}$-line.*

Proof : Let P_q denote the supporting plane of q. Choose any $z \in P_q - q$. Then it follows from the definition of Q_z that $q \subset Q_z$. Moreover, two different $\mathbb{O}$-lines meet along a linear subspace of $\mathbb{OP}^2$ [Za, proposition 3.2]. Since P_q is not contained in $\mathbb{OP}^2$, there is no other $\mathbb{O}$-line containing q. $\square$

The Chow ring of the Cayley plane was computed in [IM]. The Hasse diagram of the Schubert varieties is the following, classes are indexed by their degree, that is, the codimension of the corresponding Schubert variety:



Among the Schubert classes some are particularly remarkable. The class of an $\mathbb{O}$-line is σ_8. Moreover the Cayley plane has two families of maximal linear spaces, some of dimension five whose class is σ''_{11}, and some of dimension four whose class is σ'_{12}. Note also that Poincaré duality is given by the obvious symmetry of the Hasse diagram.

Let σ'_4 be the Schubert class which is Poincaré dual to σ'_{12}. The Hasse diagram shows that the Chow ring $A^*(\mathbb{OP}^2)$ is generated by H, σ'_4 and σ_8 (proposition 2.2 will give a stronger result), and the multiplication

table of its Schubert cells is completely determined by the Chevalley formula, Poincaré duality and the two formulas

$$(\sigma_4')^2 = \sigma_8 + \sigma_8' + \sigma_8'', \qquad \sigma_4'\sigma_8 = \sigma_{12}'.$$

An easy consequence is a presentation of $A^*(\mathbb{OP}^2)$ as a quotient of a polynomial algebra over $\mathbb{Z}$:

Proposition 2.2 *Let $\mathcal{H} = \mathbb{Z}[h,s]/(3hs^2 - 6h^5s + 2h^9, s^3 - 12h^8s + 5h^{12})$. Mapping h to H and s to σ_4' yields an isomorphism of graded algebras*

$$\mathcal{H} \simeq A^*(\mathbb{OP}^2).$$

Proof : Let $\sigma = \sigma_4'$. Using the Chevalley formula, we get successively that

$$\begin{array}{rcl}
\sigma_4'' & = & H^4 - \sigma \\
\sigma_5' & = & H\sigma \\
\sigma_5'' & = & -2H\sigma + H^5 \\
\sigma_6'' & = & -2H^2\sigma + H^6 \\
\sigma_6' & = & 3H^2\sigma - H^6 \\
\sigma_7'' & = & -2H^3\sigma + H^7 \\
\sigma_7' & = & 5H^3\sigma - 2H^7 \\
\sigma_8 & = & \sigma^2 + 2H^4\sigma - H^8 \\
\sigma_8' & = & -\sigma^2 + 3H^4\sigma - H^8 \\
\sigma_8'' & = & \sigma^2 - 5H^4\sigma + 2H^8
\end{array}$$

The last three identities have been found taking into account the fact $\sigma^2 = \sigma_8 + \sigma_8' + \sigma_8''$ [IM, (16), p.11]. Computing a hyperplane section of these cells, we get the relation $(\sigma_8 - \sigma_8' + \sigma_8'')H = 0$, namely

$$3H\sigma^2 - 6H^5\sigma + 2H^9 = 0. \tag{2}$$

In the following, we compute over the rational numbers and use this relation to get rid of the terms involving σ^2. We get

$$\begin{array}{rcl}
\sigma_9' & = & 4H^5\sigma - 5/3H^9 \\
\sigma_9'' & = & -3H^5\sigma + 4/3H^9 \\
\sigma_{10}' & = & 4H^6\sigma - 5/3H^{10} \\
\sigma_{10}'' & = & -7H^6\sigma + 3H^{10} \\
\sigma_{11}' & = & 4H^7\sigma - 5/3H^{11} \\
\sigma_{11}'' & = & -11H^7\sigma + 14/3H^{11} \\
\sigma_{12}'' & = & -11H^8\sigma + 14/3H^{12} \\
\sigma_{12}' & = & 15H^8\sigma - 19/3H^{12}
\end{array}$$

Using the relation $\sigma_4'.\sigma_8 = \sigma_{12}'$ [IM, proposition 5.2], we therefore get the relation in degree 12:

$$\sigma^3 - 12H^8\sigma + 5H^{12} = 0. \tag{3}$$

This implies $26H^9\sigma = 11H^{13}$, which we use to eliminate σ and get $H^i = 78\sigma_i$ for $13 \leq i \leq 16$.

Therefore, we see that there is indeed a morphism of algebras $f : \mathcal{H} \to A^*(\mathbb{OP}^2)$ mapping h to H and s to σ_4'. This map is surjective up to degree 8, by the previous computation of the Schubert classes. In the remaining degrees, one can similarly express the Schubert cells as integer polynomials in H and σ_4', proving that f is surjective. We claim that $\mathcal{H}$ is a free $\mathbb{Z}$-module of rank 27. Therefore, f must also be injective and it is an isomorphism.

Let us check that $\mathcal{H}$ is indeed a free $\mathbb{Z}$-module of rank 27. Let $\mathbb{Z}[h,s]_d \subset \mathbb{Z}[h,s]$ (resp. $\mathcal{H}_d \subset \mathcal{H}$) denote the degree-$d$ part (where of course h has degree 1 and s has degree 4). Since there are no relations in degree d for $0 \leq d \leq 8$, $\mathcal{H}_d$ is a free $\mathbb{Z}$-module. For $9 \leq d \leq 12$, since $\mathcal{H}_d = (\mathbb{Z}.h^{d-8}s^2 \oplus \mathbb{Z}.h^{d-4}s \oplus \mathbb{Z}.h^d)/(3h^{d-8}s^2 - 6h^{d-4}s + 2h^d)$ and $3, 6, 2$ are coprime, $\mathcal{H}_d$ is free.

Let $13 \leq d \leq 16$. Eliminating $h^{d-12}s^3$, it comes that $\mathcal{H}_d = (\mathbb{Z}.h^{d-8}s^2 \oplus \mathbb{Z}.h^{d-4}s \oplus \mathbb{Z}.h^d)/(3h^{d-8}s^2 - 6h^{d-4}s + 2h^d, 26h^{d-4}s - 11h^d)$. Since the (2×2)-minors of the matrix

$$\begin{pmatrix} 3 & -6 & 2 \\ 0 & 26 & -11 \end{pmatrix}$$

are 78, 33, 118, and are therefore coprime, the $\mathbb{Z}$-module $\mathcal{H}_d$ is again free.

Finally, in degree 17, the relations are

$$\begin{cases} & & 3h^9s^2 & -6h^{13}s & +2h^{17} & = & 0 \\ & 3h^5s^3 & -6h^9s^2 & +2h^{13}s & & = & 0 \\ 3hs^4 & -6h^5s^3 & +2h^9s^2 & & & = & 0 \\ & h^5s^3 & -12h^{13}s & +5h^{17} & & = & 0 \\ hs^4 & -12h^9s^2 & +5h^{13}s & & & = & 0 \end{cases}.$$

Since the determinant of the matrix

$$\begin{pmatrix} 0 & 0 & 3 & -6 & 2 \\ 0 & 3 & -6 & 2 & 0 \\ 3 & -6 & 2 & 0 & 0 \\ 0 & 1 & 0 & -12 & 5 \\ 1 & 0 & -12 & 5 & 0 \end{pmatrix}$$

is 1, $\mathcal{H}_{17} = 0$ and we are done. $\square$

2.3.2 The Freudenthal variety

The other exceptional minuscule homogeneous variety can be interpreted as the *twisted cubic over the exceptional Jordan algebra*. Consider the *Zorn algebra*

$$\mathcal{Z}_2(\mathbb{O}) = \mathbb{C} \oplus \mathcal{J}_3(\mathbb{O}) \oplus \mathcal{J}_3(\mathbb{O}) \oplus \mathbb{C}.$$

One can prove that there exists an action of E_7 on that 56-dimensional vector space (see [Fr]). Then the closed E_7-orbit inside $\mathbb{P}\mathcal{Z}_2(\mathbb{O})$ is the *Freudenthal variety* E_7/P_7. It was studied extensively by Freudenthal, and more recently in [KY] through a slightly different point of view. It can be constructed explicitely as the closure of the set of elements of the form $[1, X, \mathtt{Com}(X), \det(X)]$ in $\mathbb{P}\mathcal{Z}_2(\mathbb{O})$, where X belongs to $\mathcal{J}_3(\mathbb{O})$ and its comatrix $\mathtt{Com}(X)$ is defined by the usual formula for order three matrices, so that $X\mathtt{Com}(X) = \det(X)I$.

For future use we notice the following two geometric properties of the low-degree rational curves in the Freudenthal variety:

Lemma 2.3 *A general conic on the Freudenthal variety is contained in a unique maximal quadric.*

Proof : Let q be such a conic. We may suppose that q passes through the point $[1, 0, 0, 0]$, and we let $[0, Y, 0, 0]$ be the tangent direction to q at $[1, 0, 0, 0]$. A general point in q is of the form $[1, X, \mathtt{Com}(X), \det(X)]$.

Now the supporting plane of q is the closure of the set of points of the form $[1, sX+tY, s\mathtt{Com}(X), s\det(X)]$. To belong to the Freudenthal variety, such a point must verify the condition $(sX+tY)s\mathtt{Com}(X) = s\det(X)I$, hence $tY\mathtt{Com}(X) = (1-s)\det(X)I$ if $s \neq 0$. But q being smooth cannot verify any linear condition, so we must have $\det(X) = 0$ and $Y\mathtt{Com}(X) = 0$. This means that X has rank at most two, in fact exactly two by the genericity of q. So up to the action of E_6 we may suppose that

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathtt{Com}(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $Z = \mathtt{Com}(X)$ is a rank one matrix, hence defines a point of $\mathbb{OP}^2$, and since $YZ = 0$, Y has to be contained in the orthogonal (with respect to the quadratic form $M \mapsto \mathrm{tr}(M^2)$) Σ_Z to the affine tangent space of $\mathbb{OP}^2$ at $[Z]$. Finally, q must be contained in $\mathbb{C} \oplus \Sigma_Z \oplus \mathbb{C}Z \simeq \mathbb{C}^{12}$, whose intersection with the Freudenthal variety is a maximal ten dimensional quadric uniquely defined by q. This proves the claim. $\square$

Lemma 2.4 *Through three general points of the Freudenthal variety, there is a unique twisted cubic curve.*

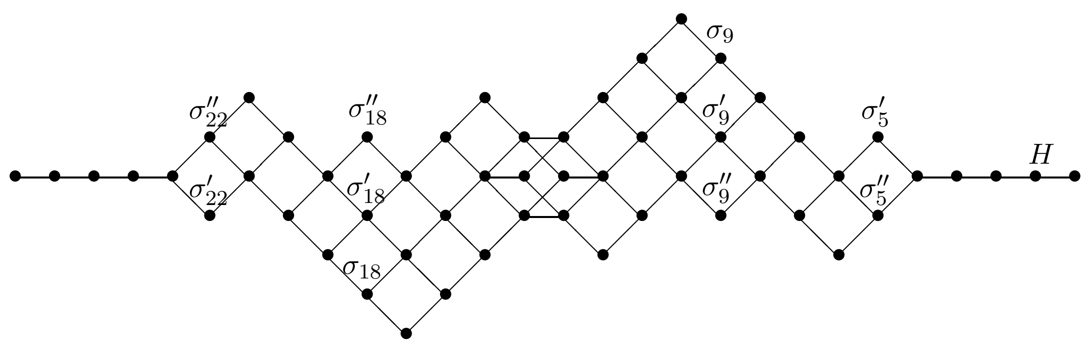
Proof : Let p_0, p_∞ be two of the three points. Because of the Bruhat decomposition, we may suppose that p_0 defines a highest weight line, and p_∞ a lowest weight line in $\mathcal{Z}_2(\mathbb{O})$. Then their common stabilizer in E_7 contains a copy of E_6, and the restriction of the E_7-module $\mathcal{Z}_2(\mathbb{O})$ to E_6 decomposes as $\mathbb{C}\oplus\mathcal{J}_3(\mathbb{O})\oplus\mathcal{J}_3(\mathbb{O})\oplus\mathbb{C}$. Otherwise said, we may suppose that in this decomposition, $p_0=[1,0,0,0]$ and $p_\infty=[0,0,0,1]$.

Now our third generic point is of the form $[1, M, \mathtt{Com}(M), \det(M)]$ with $\det(M)\neq 0$. The twisted cubic $C_0 := \{[t^3, t^2uM, tu^2\mathtt{Com}(M), u^3\det(M)] : [t,u]\in\mathbb{P}^1\}$ obviously passes through p_0, p_1, p_∞; let us prove that it is the only such curve.

For simplicity we denote the Freudenthal variety by X. Let C be a rational curve of degree three passing through p_0, p_1, p_∞ and let us proove it must be C_0. First we notice that C must be irreducible, since otherwise there would be a line or a conic through p_0 and p_∞, yielding a contradiction with $\widehat{T_{p_0}X}\cap\widehat{T_{p_\infty}X}=\{0\}$.

So let $T_i := \widehat{T_{p_i}C}\subset\mathcal{Z}_2(\mathbb{O})$ for $i\in\{0,\infty\}$. Since C is a twisted cubic, the linear span S of $\widehat{C}$ in $\mathcal{Z}_2(\mathbb{O})$ satisfies $S=T_0\oplus T_\infty$. So $p_1=[1,M,\mathtt{Com}(M),\det(M)]\in T_0\oplus T_\infty$. Now, T_0 is included in $(*,*,0,0)$ and contains $(1,0,0,0)$, and similarly for T_∞. It follows that $T_0=\{(\lambda,\mu M,0,0):\lambda,\mu\in\mathbb{C}\}$ and $T_\infty=\{(0,0,\lambda\mathtt{Com}(M),\mu):\lambda,\mu\in\mathbb{C}\}$. Therefore, $S=\{(\lambda,\mu M,\nu\mathtt{Com}(M),\omega):\lambda,\mu,\nu,\omega\in\mathbb{C}\}$, and since $X\cap\mathbb{P}S=C_0$, we get $C=C_0$. $\square$

The Chow ring of the Freudenthal variety was recently computed in [NS] with the help of a computer. The Hasse diagram of Schubert classes is the following:



We did not indicate all the Schubert cells in this diagram, but we use the same convention as for the Cayley plane: when there are several Schubert cells of the same codimension c, we denote them $\sigma''_c, \sigma'_c, \sigma_c$, starting with the lowest cell on the diagram up to degree 13, and with the highest one from degree 14. Poincaré duality is given by the obvious central symmetry of this diagram.

As we will see, the Chow ring is generated by the hyperplane section H and two Schubert classes of degree 5 and 9 (in accordance with (1)). We choose the Schubert classes σ'_5 and σ_9. The results of Nikolenko and Semenov can be stated as follows:

$$
\begin{aligned}
(\sigma'_5)^2 &= 2\sigma'_9H,\\
\sigma'_5\sigma''_5 &= (\sigma_9+\sigma'_9+2\sigma''_9)H,\\
(\sigma''_5)^2 &= (3\sigma'_9+\sigma''_9)H,\\
\sigma_9\sigma'_5 &= \sigma_{14}+2\sigma'_{14}+2\sigma''_{14},\\
\sigma'_9\sigma'_5 &= 3\sigma_{14}+4\sigma'_{14}+4\sigma''_{14},\\
\sigma''_9\sigma'_5 &= 2\sigma_{14}+3\sigma'_{14}+2\sigma''_{14},\\
\sigma_9\sigma''_5 &= \sigma_{14}+3\sigma'_{14}+3\sigma''_{14},\\
\sigma'_9\sigma''_5 &= 4\sigma_{14}+6\sigma'_{14}+5\sigma''_{14},\\
\sigma''_9\sigma''_5 &= 3\sigma_{14}+3\sigma'_{14}+3\sigma''_{14},\\
(\sigma_9)^2 &= 2\sigma_{18}+2\sigma'_{18},\\
(\sigma'_9)^2 &= 4\sigma_{18}+10\sigma'_{18}+6\sigma''_{18},\\
(\sigma''_9)^2 &= 2\sigma_{18}+4\sigma'_{18}+2\sigma''_{18},\\
\sigma_9\sigma'_9 &= 2\sigma_{18}+4\sigma'_{18}+3\sigma''_{18},\\
\sigma'_9\sigma''_9 &= 3\sigma_{18}+6\sigma'_{18}+4\sigma''_{18},\\
\sigma_9\sigma''_9 &= 3\sigma'_{18}+2\sigma''_{18}
\end{aligned}
$$

These formulas, plus the Chevalley formula and Poincaré duality, completely determine the multiplication table of Schubert classes in E_7/P_7. As for the case of the Cayley plane, we deduce a presentation of $A^*(E_7/P_7)$.

Theorem 2.5 *Let* $\mathcal{H} = \mathbb{Z}[h,s,t]/(s^2 - 10sh^5 + 2th + 4h^{10}, 2st - 12sh^9 + 2th^5 + 5h^{14}, t^2 + 922sh^{13} - 198th^9 - 385h^{18})$*. Mapping h to H, s to σ_5' and t to σ_9 yields an isomorphism of graded algebras*

$$\mathcal{H} \simeq A^*(E_7/P_7).$$

Proof : The fact that the displayed relations are relations in the Chow ring follows from the previous formulas and an expression of the Schubert cells as polynomials in the generators similar to that we did for the proof of proposition 2.2.

We therefore have a morphism $f : \mathcal{H} \to A^*(E_7/P_7)$ mapping h to H, s to σ_5' and t to σ_9; the fact that f is surjective can be read on the Hasse diagram as for proposition 2.2, except in degree 14. But in this degree, we note that $\sigma_{14} + \sigma_{14}' = \sigma_{13}.H, \sigma_{14} + \sigma_{14}'' = \sigma_{13}'.H$ and $\sigma_{14}' + \sigma_{14}'' = \sigma_{13}''.H$ belong to the image of f. This does not imply that $A^{14}(E_7/P_7) \subset \mathtt{Im}(f)$, but the surjectivity of f follows from the equality $\sigma_9'.\sigma_5' = \sigma_{14} + 2\sigma_{14}' + 2\sigma_{14}''$.

The injectivity of f will again follow from the fact that $\mathcal{H}$ is a free module of rank 56. This is a lengthy computation; we only give some indications to the reader for the relevant degrees. Our strategy is that we use the first and the third relation to get rid of monomials involving s^2 or t^2, and then use Gauss elimination to prove that $\mathcal{H}$ is free. We denote $\mathcal{H}_i$ the component of $\mathcal{H}$ of degree i. So for example, in degree 14, $\mathcal{H}_{14}$ is generated as a module by h^{14}, h^9s, h^5t, st, which satisfy the relation $2st - 12h^9s + 2h^5t + 5h^{14} = 0$. Since $2, -12, 2, 5$ are comprime, $\mathcal{H}_{14}$ is a free $\mathbb{Z}$-module. For $\mathcal{H}_{19}$, the second relation, multiplied by h^5 and s, gives

$$\begin{cases} 2h^5st - 12h^{14}s + 2h^{10}t + 5h^{19} = 0 \\ 22h^5st + 3573h^{14}s - 776h^{10}t - 1492h^{19} = 0. \end{cases}$$

After Gauss elimination, we find that this is equivalent to

$$\begin{cases} 3094h^5st - 39h^{14}s - 8962h^{10}t = 0 \\ 1238h^5st - 18h^{14}s - 358h^{10}t + h^{19} = 0, \end{cases}$$

therefore $\mathcal{H}_{19}$ is again a free module. Similaly, in degree 23, the relations are :

$$\begin{cases} 1238h^9st - 18h^{18}s - 358h^{14}t + h^{23} = 0 \\ -1312h^9st + 52h^{18}s - h^{14}t = 0 \\ 5586h^9st - 221h^{18}s = 0. \end{cases}$$

Finally, in degree 28, we multiply the second relation by h^{14}, h^9s, h^5t, st, and get :

$$\begin{cases} 2h^{14}st - 12h^{23}s + 2h^{19}t + 5h^{28} = 0 \\ 22h^{14}st - 776h^{19}t + 3573h^{23}s - 1492h^{28} = 0 \\ 384h^{14}st + 4089h^{19}t - 19514h^{23}s + 8146h^{28} = 0 \\ 7929h^{14}st - 114572h^{19}t + 521102h^{23}s - 217624h^{28} = 0. \end{cases}$$

Since the determinant of this system is 1, $\mathcal{H}_{28}$ is the trivial $\mathbb{Z}$-module. $\square$

2.4 Quivers and Poincaré duality

The archetypal minuscule homogeneous variety is the Grassmannian, whose Schubert varieties are indexed by partitions whose Ferrers diagram are contained in a fixed rectangle. In particular, Poincaré duality for Schubert classes is easily visualized: it associates to such a partition its complementary partition inside the rectangle.

In this section, we argue that there exists a general very convenient way to visualize Poincaré duality for any minuscule or cominuscule homogeneous variety $X = G/P$. This was first observed in [Pe2]. The main idea is to associate to X a quiver Q_X, which when X is a Grassmannian will be the rectangle we just mentionned.

We start with X any rational homogeneous space and a reduced expression for w_X, the longest element in W_X – say $w_X = s_{\beta_1} \cdots s_{\beta_N}$ where $N = \dim(X)$. An important point here is that in the (co)minuscule case, this reduced expression is unique up to commutation relations (see [St]).

Definition 2.6

- *For β a simple root, let $m_X(\beta)$ be the number of occurences of β in the reduced expression $w_X = s_{\beta_1} \cdots s_{\beta_N}$ ($m_X(\beta) = \#\{j : \beta_j = \beta\}$).*

- *For (β, i) such that $1 \leq i \leq m_X(\beta)$, let $r(\beta, i)$ denote the integer j such that $\beta_j = \beta$ and $\#\{k \leq j : \beta_k = \beta\} = i$. If $i > m_X(\beta)$, set $r(\beta, i) = +\infty$. Set also $r(\beta, 0) = 0$.*

- *Let Q_X be the quiver whose set of vertices is the set of pairs (β, i), where β is a simple root and $1 \leq i \leq m_X(\beta)$ and whose arrows are given as follows. There is an arrow from (β, i) to (γ, j) if $\langle \gamma^\vee, \beta \rangle \neq 0$ and $r(\gamma, j-1) < r(\beta, i) < r(\gamma, j) < r(\beta, i+1)$.*

Beware that this definition is slightly different from that given in [Pe2], but the corresponding quivers can easily be recovered one from the other. For X (co)minuscule, since the commutation relations do not change the quiver (to check this, it is enough to check that the quiver does not change when one commutes two commuting reflexions), Q_X is *uniquely defined*.

Another important property is that it is *symmetric*. Indeed, recall that the Poincaré duality on X is defined by the involution

$$w \mapsto w_0 w w_0 w_X, \quad w \in W_X.$$

In particular, since w_X defines the fundamental class in X, its Poincaré dual is the class of a point, so $w_0 w_X$ must be an involution. Thus $w_X^{-1} = w_0 w_X w_0^{-1}$. Now, if $w_X = s_{\beta_1} \cdots s_{\beta_N}$ is any reduced expression, it follows that $w_X = s_{w_0(\beta_N)} \cdots s_{w_0(\beta_1)}$ is also a reduced expression. Since $w_0(\beta_j) = \iota(\beta_j)$, where ι is the Weyl involution on the Dynkin diagram, it follows that the involution $(\beta, k) \mapsto i_X(\beta, k) = (\iota(\beta), m_X(\beta) + 1 - k)$ induces an arrow-reversing automorphism of the quiver Q_X.

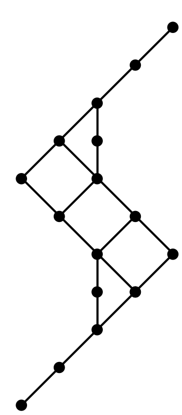
More generally, any $w \in W_X$ is given by a subexpression of some reduced expression of w_X, and again its reduced decomposition is unique up to commutation relations. We can therefore define by the same recipe a unique quiver Q_w, which is a subquiver of $Q_{w_X} = Q_X$. We think of these quivers as combinatorial tools generalizing the (strict) partitions parametrizing Schubert subvarieties in (isotropic) Grassmannians.

Example 2.7 *(ı) For a Grassmannian $X = \mathbb{G}(p, n)$, the quiver Q_X is the $p \times (n-p)$-rectangle. The set W_X can be identified with the set of partitions contained in this rectangle, and the quiver Q_w, for any $w \in W_X$, is the complement in the rectangle of the partition defining $X(w)$ (see [Pe2] and examples 3.26 and 3.30 below).*

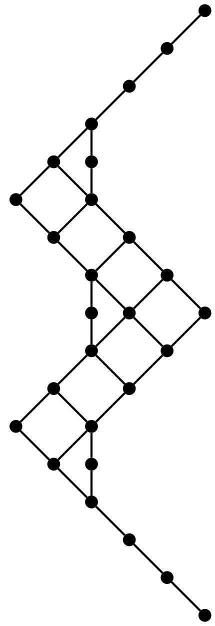
(ıı) Let $X = E_6/P_1$ be the Cayley plane. A symmetric reduced expression for w_X is given by

$$w_X = s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_1} s_{\alpha_6} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_2} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1}.$$

The quiver Q_X is the following (all arrows are going down).



(ııı) Let $X = E_7/P_7$ be the Freudenthal variety. Then Q_X is the following (again with all arrows going down).



For the quivers of the remaining classical (co)minuscule varieties we refer to [Pe2].

To avoid confusion will always draw our quivers vertically as above, contrary to Hasse diagrams which we draw horizontally. Nevertheless there is a strong connexion between quivers and Hasse diagrams, at least in the (co)minuscule cases. since we will not need it in hte sequel we just state the following result:

Proposition 2.8 *Let $X = G/P$ be a cominuscule variety with base point o. Then the quiver Q_X coincides with the Hasse diagram of the Fano variety $F_o \subset \mathbb{P}T_oX$ of lines through o.*

For example, if $X = \mathbb{G}(k,n)$, the Fano variety of lines through o is $F_o = \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$. These two projective spaces can be interpreted as the two sides of the rectangular quiver Q_X.

For any (co)minuscule homogeneous variety X, the quiver Q_X has a natural partial order given by $i \preccurlyeq j$ if there exists an oriented path from j to i (see [Pe2] for more details). This induces a partial order on each of the subquivers Q_w.

Definition 2.9 The *peaks* of Q_w are its maximal elements. We denote by $p(Q_w)$ the set of peaks of Q_w.

The fact that, for (co)minuscule homogeneous spaces, the Bruhat order is generated by simple reflections (see for example [LMS]) implies that the subquivers Q_w of Q_X corresponding to the Schubert subvarieties $X(w)$ are obtained from Q_X inductively by removing peaks. In other words, they are the Schubert subquivers of Q_X, according to the following definition:

Definition 2.10 A *Schubert subquiver* of Q_X is a full subquiver of Q_X whose set of vertices is an order ideal.

Note that w itself can easily be recovered from Q_w: simply remove from the reduced expression of w_X the reflexions corresponding to the vertices removed from Q_X to obtain Q_w.

In the more general case where X is any rational homogeneous space and we have chosen a reduced expression $\widetilde{w}_X$ of w_X giving a quiver $Q_{\widetilde{w}_X}$ depending on $\widetilde{w}_X$, we can still define the Schubert subquivers of $Q_{\widetilde{w}_X}$. As in the (co)minuscule case, these subquivers correspond to Schubert subvarieties in X. We have the following fact:

Fact 2.11 *Let X be any homogeneous variety and Q_u and Q_v two Schubert subquivers of $Q_{\widetilde{w}_X}$ corresponding to the Schubert varieties $X(u)$ and $X(v)$.*

($\imath$) If $Q_u \subset Q_v$ then $X(u) \subset X(v)$.

($\imath\imath$) If X is (co)minuscule, the converse is true.

Proof : The first part is clear. The second one comes from the fact that any Schubert subvariety is obtained inductively by removing peaks. $\square$

Now we come to Poincaré duality for X (co)minuscule. The involution i_X on Q_X induces an involution on the set of subquivers attached to the Schubert classes. Indeed, we can let

$$Q_w \mapsto Q_{i_X(w)} = i_X(Q_X - Q_w).$$

This is well defined since i_X completely reverses the partial order on Q_X: thus the set of vertices of $Q_X - Q_w$ is mapped by i_X to an order ideal.

Proposition 2.12 *The Schubert classes $[X(w)]$ and $[X(i_X(w))]$ are Poincaré dual in $A^*(X)$.*

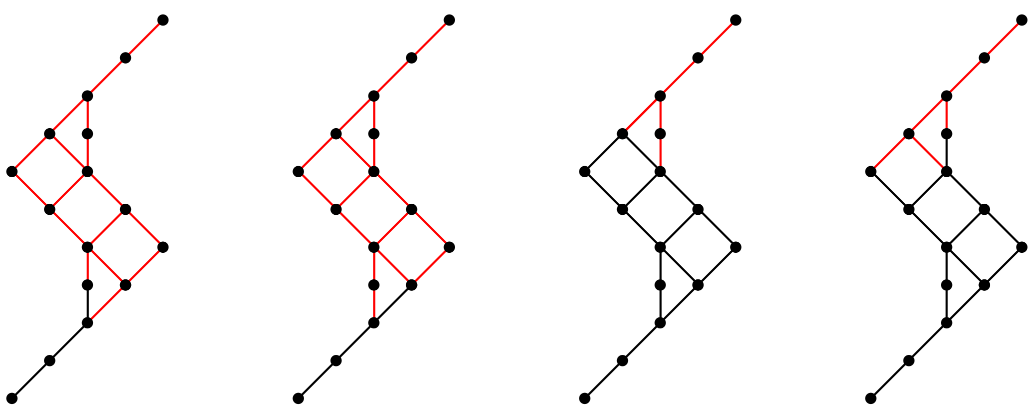
Proof : Let $w \in W_X$. There exists a symmetric reduced expression $w_X = s_{\beta_1} \cdots s_{\beta_N}$ such that $s_{\beta_{k+1}} \cdots s_{\beta_N}$ is a reduced expression for w. Since $i(\beta) = -w_0(\beta)$, the element in W_X defining the Schubert class which is Poincaré dual to $\sigma(w)$ is

$$w^* = s_{i(\beta_{k+1})} \cdots s_{i(\beta_N)} w_X = s_{\beta_{N-k+1}} \cdots s_{\beta_N}.$$

This is nothing else than $i_X(w)$. $\square$

Example 2.13 *($\imath$) For a Grassmannian, we recover the fact that the Poincaré duality is given by the complementarity of partitions in the corresponding rectangle. The previous proposition is a generalization of this fact.*

($\imath\imath$) Let X be the Cayley plane. Consider the Schubert classes $\sigma(v) = \sigma'_{12}$ and $\sigma(w) = \sigma''_{12}$. We have reduced expressions $v = s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$ and $w = s_{\alpha_2}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$. The Poincaré duals are $w^* = i_X(w) = s_{\alpha_3}s_{\alpha_4}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$, and $v^* = i_X(v) = s_{\alpha_2}s_{\alpha_4}s_{\alpha_5}s_{\alpha_1}s_{\alpha_6}s_{\alpha_3}s_{\alpha_4}s_{\alpha_2}s_{\alpha_5}s_{\alpha_4}s_{\alpha_3}s_{\alpha_1}$. The corresponding quivers are given by the following pictures, where on the left (resp. right) in black we have the quivers Q_v and Q_w (resp. Q_{v^*} and Q_{w^*}) and in red their complements in Q_X. This means that an arrow between two vertices of the subquiver is drawn in black, and the other arrows are drawn in red. The same convention will be used in all the article.



Q_w $\qquad$ Q_v $\qquad$ Q_{w^*} $\qquad$ Q_{v^*}

3 From classical to quantum invariants

In this section, we give a unified presentation of some results of A. Buch, A. Kresch and H. Tamvakis [BKT], according to which the Gromov-Witten invariants of (co)minuscule homogeneous varieties under some group G, can be computed as classical invariants on some other G-homogeneous varieties. In particular we extend these results to the exceptional cases (corollary 3.28).

3.1 Gromov-Witten invariants

First we briefly recall the definition of the quantum Chow ring of a rational homogeneous variety $X = G/P$, where P is a maximal parabolic subgroup of G. We refer to [FP] for more details.

The small (in the following, this adjective will be skipped) quantum Chow ring $QA^*(G/P)$ is the free $\mathbb{Z}[q]$-module over the Schubert classes in $A^*(G/P)$, with a product given by the following formula:

$$[X(u)] * [X(v)] = \sum_{d \geqslant 0} q^d \sum_{w \in W_X} I_d([X(u)], [X(v)], [X(w)])[X(w^*)].$$

The coefficients $I_d([X(u)], [X(v)], [X(w)])$ in this formula are the degree d *Gromov-Witten invariants*, which are defined as intersection numbers on the moduli space of stable curves. In the homogeneous setting, such an invariant counts the number of pointed maps $f : \mathbb{P}^1 \to G/P$ of degree d, such that $f(0) \in g \cdot X(u)$, $f(1) \in g' \cdot X(v)$ and $f(\infty) \in g'' \cdot X(w)$ for three general elements g, g', g'' of G (see [FP], Lemma 13). In fact $I_d([X(u)], [X(v)], [X(w)])$ can be non zero only when

$$\operatorname{codim}(X(u)) + \operatorname{codim}(X(v)) + \operatorname{codim}(X(w)) = \dim(X) + dc_1(X).$$

The quantum product makes of $QA^*(X)$ a commutative and associative graded ring, q being given the degree $c_1(X)$ and the Schubert classes their codimensions, as in the usual Chow ring.

3.2 Lines and quantum cohomology

The associativity of the quantum Chow ring $QA^*(G/P)$ is a very strong property, and a nice consequence is that the quantum product can often be completely determined by the classical product and a small list of quantum multiplications.

We begin with a simple observation which gives almost for free such a quantum product, and makes the link with the discussion of the index in section 2.

Recall that we denoted by F the Fano variety of lines in $X = G/P \subset \mathbb{P}V$, with respect to its minimal homogeneous embedding. Also we denoted by $F_o \subset F$ the closed subscheme of lines through the base point o. Mapping such a line to its tangent direction at o yields a closed embedding of F_o in $\mathbb{P}T_oX$. In the following proposition, the degree of F_o is understood with respect to this embedding.

Proposition 3.1 *The quantum power of the hyperplane class H of exponent $c_1(X)$ is*

$$H^{*c_1(X)} = H^{c_1(X)} + \deg(F_o)q.$$

Proof : Let $L_1, L_2 \subset \mathbb{P}V$ be general linear subspaces of codimension l_1, l_2 with $l_1 + l_2 = \dim(F) - \dim(X) + 3$. We need to prove that $I_1([o], [L_1], [L_2]) = \deg(F_o)$. As we have seen, since $\operatorname{codim}(o) + \operatorname{codim}(L_1 \cap X) + \operatorname{codim}(L_2 \cap X) = \dim(F) + 3 = \dim(X) + c_1(X)$, the Gromov-Witten invariant $I_1([o], [L_1], [L_2])$ counts the number of lines in X through o meeting L_1 and L_2.

Let $\widehat{L}_i \subset V$ denote the affine cone over L_i and let $p : \widehat{T_oX} \to T_oX = \widehat{T_oX}/\ell_o$ be the natural projection, where $\ell_o \subset V$ is the line defined by o. In terms of the projective embedding $F_o \subset \mathbb{P}T_oX$, a line meets L_1 and L_2 if and only if the corresponding point lies in $L := \mathbb{P}(p(\widehat{L_1} \cap \widehat{T_oX}) \cap p(\widehat{L_2} \cap \widehat{T_oX}))$. Since $\dim F_o = \dim F - \dim X + 1 = l_1 + l_2 - 2 = \operatorname{codim}_{\mathbb{P}T_oX} L$, there are $\deg(F_o)$ such points. $\square$

In fact this proposition is true in a much larger generality. The previous proof adapts almost verbatim to any projective variety for which quantum cohomology is defined.

3.3 Quivers and Poincaré duality for the Fano variety of lines

As in the first section, let us denote by F the Fano variety of lines in $X = G/P$. Denote by I the incidence variety and by p and q the projections from I to X and F:

$$\begin{tikzcd} I \arrow[r, "p"] \arrow[d, "q"] & X \\ F & \end{tikzcd}$$

If ℓ is a point in F, we denote by L the corresponding line in X. We have $L = p(q^{-1}(\ell))$. The varieties F and I are homogeneous: if $X = G/P$ where P is associated to a simple root β, then $F = G/Q$ where Q is

associated to the simple roots linked to β in the Dynkin diagram (see [LM], Theorem 4.8; but beware that this is not true for all rational homogeneous varieties). Moreover $I = G/R$ where $R = P \cap Q$.

Remark that the fiber Z of p is isomorphic to a product of (co)minuscule homogeneous varieties under a subgroup of G. In particular there is a uniquely defined quiver $Q_Z = Q_{w_Z}$ for Z, constructed from any reduced expression of w_Z.

We have $w_I = w_X w_Z$. This can be seen as follows. By the Bruhat decomposition, $Bw_Z R$ is open in P. By the projection map to $X = G/P$, the subset $Bw_X B w_Z R/R$ of $I = G/R$ is thus mapped to $Bw_X P/P$, the open cell of X. The fiber of the base point is $p^{-1}(w_X P/P) \cap Bw_X Bw_Z R/R = (Bw_X \cap w_X P)Bw_Z R/R$, a dense open subset of $p^{-1}(w_X P/P) = w_X P/R$. Thus $Bw_X Bw_Z R/R$ is an open subset of I. Its dimension $\ell(w_X)+\ell(w_Z)$ is thus equal to the dimension $\ell(w_I)$ of I. In particular, $w_X w_Z$ is reduced and $Bw_X Bw_Z R/R$ contains $Bw_X w_Z R/R$ as an open subset. But the later must be the open cell in I, thus $w_X w_Z = w_I$ as claimed.

By the same argument, we get that $w_I = s_{\beta_1} w_F$, where β_1 denotes the unique simple root such that $l(s_{\beta_1} w) = l(w) - 1$. Now, if we choose reduced expressions $s_{\beta_1} \cdots s_{\beta_N}$ for w_X and $s_{\beta'_1} \cdots s_{\beta'_M}$ for w_Z (with $M = \dim(Z)$), we obtain a reduced expression $s_{\beta_2} \cdots s_{\beta_N} s_{\beta'_1} \cdots s_{\beta'_M}$ for w_F. This reduced expression is uniquely defined modulo commutation relations, although it is not true that w_F has a unique reduced expression modulo commutation relations. We thus get a uniquely defined quiver $Q_F = Q_{w_F}$. Of course we also have a quiver Q_I, which is deduced from Q_F by attaching a vertex (corresponding to β_1) to the highest vertex of Q_F.

As in the (co)minuscule case, the quiver Q_F is symmetric: the fact that $w_0 w_X$ and $w_0 w_F$ are involutions ensures that our reduced expression of w_F can be chosen good, in the sense of the following definition:

Definition 3.2 A good reduced expression for w_F is a reduced expression

$$w_F = s_{\gamma_1} \cdots s_{\gamma_R} = s_{\beta_2} \cdots s_{\beta_N} s_{\beta'_1} \cdots s_{\beta'_M},$$

where $w_X = s_{\beta_1} \cdots s_{\beta_N}$, $w_Z = s_{\beta'_1} \cdots s_{\beta'_M}$, $R = \dim(F)$, and $i(\gamma_k) = \gamma_{R+1-k}$.

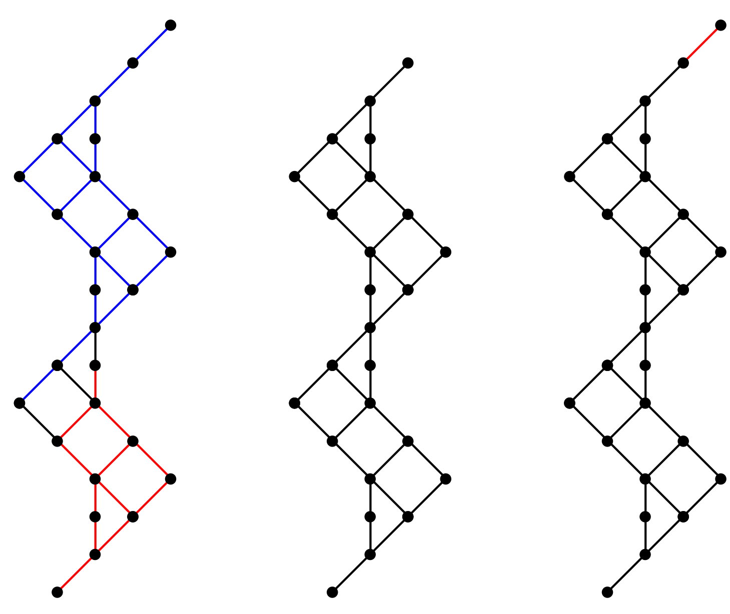
Example 3.3 *Let us fix X as in example 2.7 (ii). The element w_Z has the following reduced expression*

$$w_Z = s_{\alpha_2} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_2} s_{\alpha_4} s_{\alpha_3}$$

and we deduce for w_F a good reduced expression

$$w_F = s_{\alpha_5} s_{\alpha_4} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_1} s_{\alpha_3} s_{\alpha_4} s_{\alpha_2} s_{\alpha_6} s_{\alpha_5} s_{\alpha_4} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_4} s_{\alpha_5} s_{\alpha_6} s_{\alpha_3} s_{\alpha_4} s_{\alpha_5} s_{\alpha_2} s_{\alpha_4} s_{\alpha_3}.$$

The quivers Q_I and Q_F have the following forms:



$Q_X \subset Q_I \supset Q_Z$ $\qquad$ Q_F $\qquad$ $Q_F \subset Q_I$

where on the left picture we have drawn in blue the quiver Q_X embedded in Q_I and in red the quiver Q_Z embedded in Q_I. In the right one we have drawn in black the quiver Q_F embedded in Q_I and in red its complement. The middle picture is Q_F.

As in subsection 2.4 the symmetry of the quiver Q_F induces an involution $i_F : Q_w \mapsto Q_{i_F(w)}$ on the Schubert subquivers (recall definition 2.10 of Q_F). The same proof as that of Proposition 2.12 gives the following partial interpretation of Poincaré duality on F:

Proposition 3.4 *Let $F(w)$ be a Schubert subvariety of F such that there exists a good reduced expression $w_F = s_{\gamma_1} \cdots s_{\gamma_R}$ and an integer k such that $w = s_{\gamma_{k+1}} \cdots s_{\gamma_R}$. Then the classes $[F(w)]$ and $[F(i_F(w))]$ are Poincaré dual.*

Remark 3.5 (ı) Beware that not all Schubert varieties $F(w), w \in W_F$ satisfy the hypothesis of the proposition. This is because F is not minuscule and in consequence there may be braid relations. However, all Schubert varieties $F(w)$ associated to a Schubert subquiver $Q_F(w)$ of Q_F satisfy the property.

(ıı) We will denote by $F(u^\star)$ the Poincaré dual of $F(u)$.

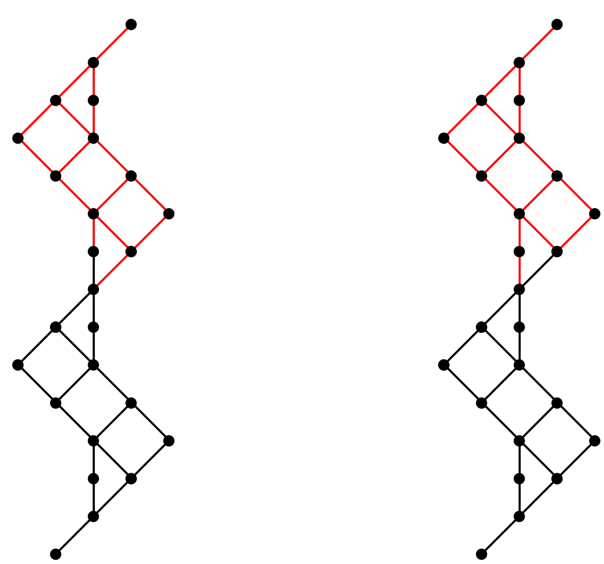
Let $X(w)$ be a Schubert variety in X. We define the Schubert variety $F(\widehat{w}) = q(p^{-1}(X(w)))$ of F. The variety $F(\widehat{w})$ parametrises the lines in X meeting $X(w)$. For any Schubert variety $X(w)$, we can choose a reduced expression $w_X = s_{\beta_1} \cdots s_{\beta_N}$ such that w has a reduced expression $s_{\beta_{k+1}} \cdots s_{\beta_N}$. Then we get for $\widehat{w}$ the reduced expression $\widehat{w} = s_{\beta_{k+1}} \cdots s_{\beta_N} w_Z = s_{\gamma_{k+1}} \cdots s_{\gamma_R}$. In particular $F(\widehat{w})$ satisfies the hypothesis of proposition 3.4.

Consider the quivers Q_X and Q_F as subquivers of Q_I: the first one is obtained by removing Q_Z from the bottom of Q_I; the second one by removing the top vertex of Q_I. The quiver of $F(\widehat{w})$ is deduced from the quiver of $X(w)$ by removing from $Q_F \subset Q_I$ the vertices of Q_X not contained in Q_w. In particular, if $X(w)$ is different from X, then

$$\mathrm{codim}_F(F(\widehat{w})) = \mathrm{codim}_X(X(w)) - 1.$$

Remark that a Schubert subvariety $F(v)$, where $v = s_{\gamma_{k+1}} \cdots s_{\gamma_R}$, is of the form $F(\widehat{w})$ for some w, if and only if the quiver $Q_{F(v)}$ contains Q_Z. In that case Q_w is the complement of Q_Z in $Q_{F(v)}$.

Example 3.6 *Let X, $X(w)$ and $X(v)$ as in example 2.13 (ıı). Then the quivers of $F(\widehat{w})$ and $F(\widehat{v})$ are the following (in black):*



$Q_{F(\widehat{w})}$ $Q_{F(\widehat{v})}$

3.4 Computing degree one Gromov-Witten invariants

In this subsection, we explain how to apply the technics of A. Buch, A. Kresch and H. Tamvakis [BKT] to calculate degree one Gromov-Witten invariants.

Lemma 3.7 *Let $X(w)$ and $X(v^*)$ be two Schubert varieties such that $X(w) \subset X(v^*)$. Then there exists an element $g \in G$ such that the intersection $X(w) \cap g \cdot X(v)$ is a reduced point.*

Proof : We proceed by induction on $a = \dim(X(v^*)) - \dim(X(w))$. If $a = 0$ this is just Poincaré duality. Assume that the result holds for $X(w')$ such that $X(w)$ is a divisor in $X(w')$. Then there exists an element $g \in G$ such that $X(w') \cap g \cdot X(v)$ is a reduced point x. Now we use the following fact on (co)minuscule Schubert varieties (cf. [LMS]):

Fact 3.8 *Any divisor $X(w)$ in $X(w')$ is a moving divisor, in particular we have*

$$X(w') = \bigcup_{h \in \mathrm{Stab}(X(w'))} h \cdot X(w).$$

We deduce from this fact that there exist an element $h \in \mathrm{Stab}(X(w'))$ such that $x \in h \cdot X(w)$. But now $h \cdot X(w)$ meets $g \cdot X(v)$ in x at least. Since $h \cdot X(w)$ is contained in $X(w')$ and $X(w') \cap g \cdot X(v)$ is the reduced point x, $h \cdot X(w)$ and $g \cdot X(v)$ meet only in x, and transversely at that point. □

Remark 3.9 Applying the lemma to the case where $X(v^*)$ is the codimension one Schubert subvariety of X, we obtain that if $X(w)$ is different from X and ℓ is a general point in $F(\widehat{w})$, then L meets $X(w)$ in a unique point. In this paragraph we will only use this version of the lemma. In section 3.6, we use it in its general formulation.

Lemma 3.10 *Let $X(u)$, $X(v)$ and $X(w)$ be three proper Schubert subvarieties of X, such that*

$$\mathit{codim}(X(u)) + \mathit{codim}(X(v)) + \mathit{codim}(X(w)) = \dim(X) + c_1(X).$$

Then for g, g' and g'' three general elements in G, the intersection $g \cdot F(\widehat{u}) \cap g' \cdot F(\widehat{v}) \cap g'' \cdot F(\widehat{w})$ is a finite set of reduced points.

Let ℓ be a point in this intersection, then the line L meets each of $g \cdot X(u)$, $g' \cdot X(v)$ and $g'' \cdot X(w)$ in a unique point and these points are in general position in L.

Proof : The codimension condition and the fact that $X(u)$, $X(v)$ and $X(w)$ are different from X imply that

$$\mathrm{codim}(F(\widehat{u})) + \mathrm{codim}(F(\widehat{v})) + \mathrm{codim}(F(\widehat{w})) = \dim(F).$$

In particular, the first part of the proposition follows from Bertini's theorem (see [Kl]). Furthermore, by Bertini again, we may assume that any ℓ in the intersection is general in $g \cdot F(\widehat{u})$, $g' \cdot F(\widehat{v})$ and $g'' \cdot F(\widehat{w})$. In particular, L meets each of $g \cdot X(u)$, $g' \cdot X(v)$ and $g'' \cdot X(w)$ in a unique point by the previous lemma. Finally, the stabiliser of ℓ acts transitively on L and by modifying g, g' and g'' by elements in this stabiliser me may assume that the points are in general position in L. □

Corollary 3.11 *Let $X(u)$, $X(v)$ and $X(w)$ be three proper Schubert subvarieties of X. Suppose that the sum of their codimensions is $\dim(X) + c_1(X)$. Then*

$$I_1([X(u)], [X(v)], [X(w)]) = I_0([F(\widehat{u})], [F(\widehat{v})], [F(\widehat{w})]).$$

Proof : The image of any morphism counting in the invariant $I_1([X(u)], [X(v)], [X(w)])$ is a line ℓ in the intersection $g \cdot F(\widehat{u}) \cap g' \cdot F(\widehat{v}) \cap g'' \cdot F(\widehat{w})$ for general elements g, g' and g'' in G. The preceding lemma implies that there is a finite number $I_0([F(\widehat{u})], [F(\widehat{v})], [F(\widehat{w})])$ of such lines and that given such a line, there exists a unique morphism from $\mathbb{P}^1$ to L with 0, 1 and ∞ sent to the intersection of L with $g \cdot X(u)$, $g' \cdot X(v)$ and $g'' \cdot X(w)$. □

In order to make some computations on F we prove the following lemma.

Lemma 3.12 *In any homogeneous variety Y, the intersection product $[Y(u)]\cdot[Y(v)]$ vanishes if and only if $Y(v)$ does not contain $Y(u^\star)$ where $\star$ is the Poincaré duality on Y.*

Proof : The product $[Y(u)]\cdot[Y(v)]$ does not vanish if and only there exists a sequence of codimension one Schubert subvarieties $H_1,\ldots,H_a$, with $a=\dim(Y)-\mathrm{codim}(Y(u))-\mathrm{codim}(Y(v))$, such that the product $[H_1]\cdots[H_a]\cdot[Y(u)]\cdot[Y(v)]$ is non zero (recall that $\mathrm{Pic}(Y)$ may be bigger than $\mathbb{Z}$). But we have

$$[H_1]\cdots[H_a]\cdot[Y(v)]=\sum_w c_w^{H_1\cdots H_a}[Y(w)]$$

where the sum runs over all w such that $\dim(Y(w))=\mathrm{codim}(Y(u))$. The product $[H_1]\cdots[H_a]\cdot[Y(u)]\cdot[Y(v)]$ is non zero if and only if $c_{u^\star}^{H_1\cdots H_a}$ is non zero.

As a consequence the product $[Y(u)]\cdot[Y(v)]$ does not vanish if and only there exists a sequence of codimension one Schubert varieties $H_1,\cdots,H_a$ such that $c_{u^\star}^{H_1\cdots H_a}\neq 0$. But this is equivalent to the inclusion $Y(u^\star)\subset Y(v)$. □

Using this lemma on a (co)minuscule Schubert variety X will be easy because inclusions of Schubert varieties are equivalent to inclusions of their quivers (Fact 2.11). This is not the case for a general rational homogeneous variety and in particular for F. However we have the following lemma:

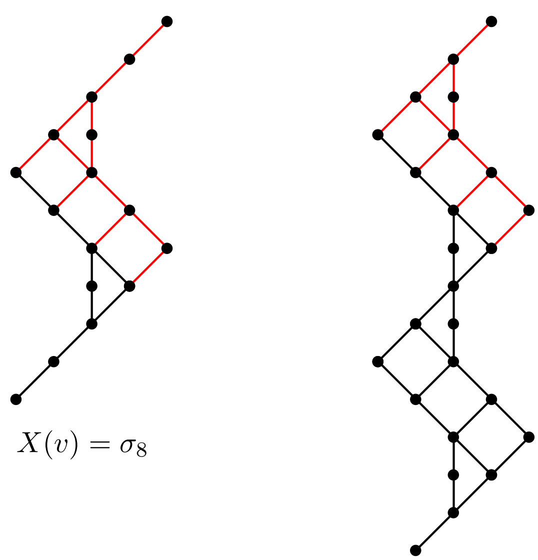
Lemma 3.13 *Let u and v in W_F such that $F(u)$ and $F(v)$ are represented by Schubert subquivers $Q_F(u)$ and $Q_F(v)$ of Q_F.*

($\imath$) If $Q_F(u)\subset Q_F(v)$ then $F(u)\subset F(v)$ (see Fact 2.11 ($\imath$)).

($\imath\imath$) Conversely, if $F(u)\subset F(v)$, then we have the inclusion $Q_F(u)\cap i_F(Q_Z)\subset Q_F(v)\cap i_F(Q_Z)$.

Proof : For ($\imath\imath$), remark that we have the equality $w_I=w_F s_{\iota(\beta_1)}$ so that we get a quiver for I by adding at the end of Q_F a vertex corresponding to $i(\beta_1)$. Adding the same vertex at the end of $i_F(Q_Z)$ gives the quiver Q_X. More generally, adding the same vertex at the end of $Q_F(u)\cap i_F(Q_Z)$ gives the quiver of $p(q^{-1}(F(u))$. In particular, if $F(u)\subset F(v)$ we must have $p(q^{-1}(F(u))\subset p(q^{-1}(F(v))$ and an inclusion of the corresponding quivers because X is minuscule. This gives the desired condition. □

Example 3.14 *We come back to example 2.7 ($\imath\imath$). Let $X(v)$ be the Schubert subvariety corresponding to the class σ_8. The quiver Q_v has the following form:*

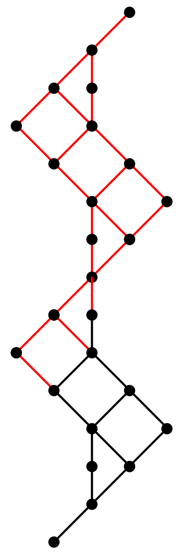


$X(v)=\sigma_8$ $\qquad$ $Q_{F(\widehat{v})}$

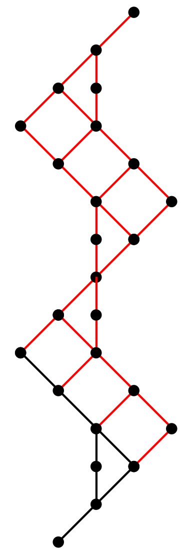
On the left we have drawn in black the quiver Q_v inside Q_X with its complement in red. On the right we have drawn in black the quiver of $F(\widehat{v})$ inside Q_F with its complement in red.

Now let $X(u)$ be any Schubert subvariety of codimension four. For degree reasons, the quantum product $X(u)*X(v)$ can only have terms in q^0 or q^1. Let us concentrate on q-terms. We need to compute $I_3([F(\widehat{u})],[F(\widehat{v})],[F(\widehat{w})])$ for w such that codim$(X(w))=16$, that is, $X(w)=\{pt\}$. By Lemma 3.13, the

Schubert variety $F(\widehat{v}^\star)$ is not contained in $F(\widehat{w})$. Indeed, both quivers are contained in $\iota_F(Q_Z)$ but we don't have $Q_F(\widehat{v}^\star) \subset Q_F(\widehat{w})$ (see the following pictures).



Quiver of $F(\widehat{w})$



Quiver of $F(\widehat{v}^\star)$

This implies thanks to lemma 3.12 that $[F(\widehat{v})]\cdot[F(\widehat{w})] = 0$ and in particular $I_0([F(\widehat{u})], [F(\widehat{v})], [F(\widehat{w})]) = 0$. We conclude that for any codimension four class τ in $A^*(X)$, we have:

$$\tau * \sigma_8 = \tau \cdot \sigma_8.$$

3.5 Some geometry of rational curves

We will give in the next section a way of computing higher Gromov-Witten invariants similar to what we did for degree one. The starting point is to find a variety playing the role of F for rational curves of degree $d > 1$. We have seen that the fact that F is homogeneous plays a crucial role in the proofs. However, the variety of degree d rational curves on X is not homogeneous for $d \geq 2$. We will introduce in this section a homogeneous variety F_d which will play the role of F for $d \geq 2$.

F.L. Zak suggested to study the following integer $d(x, y)$.

Definition 3.15

- Let $x, y \in X$. We denote $d(x, y)$ the least integer δ such that there exists a degree δ union of rational curves through x and y. This gives X the structure of a metric space.

- For $x, y \in X$, we define $Y(x, y)$ as the union of all degree $d(x, y)$ unions of rational curves through x and y. Let Y_d denote the abstract variety $Y(x, y)$, for any couple (x, y) such that $d(x, y) = d$.

- Let α denote the root defining P and let $d_{max}(X)$ denote the number of occurences of s_α in a reduced expression of w_X.

- If $d \in [0, d_{max}]$, we denote F_d the set of all $Y(x, y)$'s, for $x, y \in X$ such that $d(x, y) = d$.

The varieties F_d and Y_d are well-defined in view of the following proposition:

Proposition 3.16 *The group G acts transitively on the set of couples of points (x, y) with $d(x, y) = d$, and $d(x, y)$ takes exactly all the values between 0 and d_{max}. If $\omega \in F_d$, then the stabilizer of ω is a parabolic subgroup of G, thus giving F_d the structure of a projective variety. Moreover, Y_d and F_d are as in the*

following table :

X	d_{max}	d	F_d	Y_d
$\mathbb{G}(p,n)$	$\min(p,n-p)$		$\mathbb{F}(p-d,p+d;n)$	$\mathbb{G}(d,2d)$
$\mathbb{G}_\omega(n,2n)$	n		$\mathbb{G}_\omega(n-d,2n)$	$\mathbb{G}_\omega(d,2d)$
$\mathbb{G}_Q(n,2n)$	$\frac{n}{2}$		$\mathbb{G}_Q(n-2d,2n)$	$\mathbb{G}_Q(2d,4d)$
$\mathbb{Q}^n$	2	$d=2$	$\{\mathrm{pt}\}$	$\mathbb{Q}^n$
E_6/P_1	2	$d=2$	E_6/P_6	$\mathbb{Q}^8$
E_7/P_7	3	$d=2$	E_7/P_1	$\mathbb{Q}^{10}$
E_7/P_7	3	$d=3$	$\{\mathrm{pt}\}$	E_7/P_7

Proof : Let us denote temporarily F'_d, Y'_d the varieties in definition 3.15, and let F_d, Y_d denote the homogeneous varieties in the above array. The result of the proposition, that $F_d = F'_d$ and $Y_d = Y'_d$, will follow from facts about F_d, Y_d. If $X = G/P$, then the variety F_d is a homogeneous variety under G of the form $F_d = G/Q$ with Q a parabolic subgroup (if $F_d = \{\mathrm{pt}\}$ then $Q = G$). In particular we have an incidence variety I_d and morphisms $p_d : I_d \to X$ and $q_d : I_d \to F_d$ giving rise to the diagram:

$$\begin{tikzcd} I_d \arrow[r, "p_d"] \arrow[d, "q_d"] & X \\ F_d & \end{tikzcd}$$

If $\omega \in F_d$, we denote $Y_\omega = p_d(q_d^{-1}(\omega))$. In the following, Z_d will denote a fiber of p_d. The relevance of Y_ω for the study of degree d rational curves comes from the fact:

Proposition 3.17 *For any degree d rational curve C, there exists at least one element $\omega \in F_d$ such that $C \subset Y_\omega \simeq Y_d$. For a general curve, the point ω is unique.*

Proof : Assume first that C is irreducible. Then, this was observed in [BKT] for (isotropic) Grassmannians. Its is obvious when F_d is a point. For the Cayley plane and $d = 2$ this is Lemma 2.1. For the Freudenthal variety and $d = 2$ again, this is Lemma 2.3.

We now extend this result to the case of a reducible curve. We consider the moduli space $M_d(X)$ parametrizing rational curves of degree d in X, and the subset $M_d^Y(X)$ of curves included in Y_ω for some $\omega \in F_d$. Consider the relative moduli space $\mathcal{M}_d \to F_d$ whose fiber over $\omega \in F_d$ parametrizes the curves in Y_ω. We have a natural map $\mathcal{M}_d \to M_d(X)$, whose image is by definition $M_d^Y(X)$. Since $\mathcal{M}_d$ is proper, $M_d^Y(X)$ is closed in $M_d(X)$. We have seen above that it contains an open subset of $M_d(X)$, which is irreducible by [Th, Pe1]; therefore $M_d^Y(X) = M_d(X)$. $\square$

Fact 3.18 *There exists a unique degree d morphism $f : \mathbb{P}^1 \to Y_d$ passing through three general points of Y_d.*

Proof : This was proved in [BKT] for (isotropic) Grassmannians. For the other cases and $d = 2$, this is simply the fact that through three general points on a quadric, there exists a unique conic (the intersection of the quadric with the plane generated by the three points). Finally, for E_7/P_7 and $d = 3$, this is Lemma 2.4. $\square$

We now prove that $d(x,y)$ classifies the G-orbits in $X \times X$.

Let $(x,y) \in X \times X$ and assume $y \neq x$. Up to the action of G, we may assume that x is the base point and that y is the class of an element v in the Weyl group. Let d denote the number of occurences of the reflection s_α in a reduced decomposition of v. Since the reflections s_β for $\beta \neq \alpha$ belong to P, we may assume that the quiver Q_v of v has only peaks corresponding to the root α. So Q_v has only one maximal element, $(\alpha, d_{max}+1-d)$ (see definition 2.6); therefore $Q_v = \{q \in Q_X : q \preccurlyeq (\alpha, d_{max}+1-d)\}$.

Such subquivers are parametrized by $d \in [0, d_{max}]$, so there are at most $d_{max}+1$ orbits in $X \times X$. For $e \in [0, d_{max}]$, we denote v_e the corresponding element in W_X (so that $v = v_d$). Note also that the Schubert variety corresponding to v_e is isomorphic with Y_e. Now we will prove that $d = d(x,y)$. On the one hand, there is a $\mathbb{P}^1$ between the base point in G/P and the class of s_α, so by induction we deduce that $d(x,y) \leq d$.

On the other hand, assume there is a degree e rational curve through x and y; we will show that $e \geq d$. By proposition 3.17, there is an element $\omega \in Y_e$ such that x and y belong to Y_ω. Since the condition $x, y \in Y_\omega$ is a closed condition on ω, we may furthermore assume that ω is B-stable. Therefore, $p_e(q_e^{-1}(\omega))$ is the Schubert cell corresponding to the element $v_e \in W_X$. We thus have $v \leq v_e$ for the Bruhat order. This implies $d \leq e$.

We now conclude the proof of Proposition 3.16. Using our classification of the couples in $X \times X$, it is easy to check that for $x \neq y \in X$ and $d = d(x,y)$, there exists a unique $\omega \in F_d$ such that $x, y \in Y_\omega$. Moreover, (x,y) is a generic couple in $Y_\omega \times Y_\omega$. Therefore, if z is a generic point in Y_ω, by fact 3.18, z belongs to $Y(x,y)$, so that $Y(x,y) \supset Y_\omega$. On the other hand, if C is a union of rational curves of degree d, then by proposition 3.17, there exists β such that $Y_\beta \supset C$; since $x, y \in Y_\beta$ we deduce $\beta = \omega$ and $Y(x,y) \subset Y_\omega$. □

Remark 3.19 (I) The varieties $Y(x,y)$ are lines when $d(x,y) = 1$, so F_1 parametrizes lines in X (therefore $F_1 = F$).

(II) The variety F_2 parametrizes the maximal quadrics on X; the nodes defining F_2 are such that when we suppress them and keep the connected component of the remaining diagram containing the node that defines X, we get the weighted Dynkin diagram of a quadric. The maximal quadrics on X are then obtained as Tits shadows (see [LM]).

(III) The bounds on the degrees are the easy bounds for the vanishing of Gromov-Witten invariants. Namely, the degree d Gromov-Witten invariants all vanish as soon as $c_1(X)d + \dim(X) > 3\dim(X)$. (Actually, in type A these easy bounds are even more restrictive in general.)

We now deduce from the last two results some equalities of dimensions. Proposition 3.17 implies that the dimension of the scheme of degree d rational curves on X equals the dimension of the scheme of degree d rational curves on Y_d plus the dimension of F_d, that is, according to (5),

$$\dim(X) + d \cdot c_1(X) = \dim(F_d) + 3\dim(Y_d). \tag{4}$$

Remark 3.20 Conversely, this equality together with the irreducibility of the variety $\mathbf{Mor}_d(\mathbb{P}^1, X)$ of degree d morphisms from $\mathbb{P}^1$ to X (see for example [Th] or [Pe1]) implies that, for any morphism $f : \mathbb{P}^1 \to X$, there exists $\omega \in F_d$ such that f factors through Y_ω and that for a general morphism f, there is a finite number of such points ω. It is an easy verification that if there are more than one point ω then f is not general.

A consequence of Fact 3.18 is that the dimension of the scheme of degree d rational curves on Y_d is $3\dim(Y_d) - 3$. Since this dimension can also be computed from the index of Y_d, we get the relation

$$d \cdot c_1(Y_d) = 2\dim(Y_d). \tag{5}$$

We now derive some combinatorial properties which will be useful in [CMP]. Since the quiver of Y_d is the set of vertices under $(\alpha, d_{max} + 1 - d)$, the quiver of Y_d^* is the set of vertices not above $(\iota(\alpha), d)$, if ι denotes the Weyl involution of the simple roots. Let $\delta(u)$ denote the number of occurences of s_α in a reduced expression of u. We therefore have $X(u) \subset Y_d^*$ if and only if $d \leq \delta(u)$.

From the proof of proposition 3.16, it also follows that there is a curve of degree d through 1 and u if and only if $d \geq \delta(u)$.

Finally, we relate our integer $\delta(u)$ to an integer defined in [FW]. Recall the definition [FW, lemma 4.1] that two elements $u, v \in W/W_P$ are adjacent if there exists a reflection s such that $u = vs$. A chain between u and v is a sequence $u_0, \ldots, u_r$ such that $u \preccurlyeq u_0, u_r \preccurlyeq v^*$, and each u_i is adjacent to u_{i+1}. Such a chain has a natural degree. We then consider the following definition, suggested by [FW, theorem 9.1]:

Definition 3.21 Let $u, v \in W_X$. Let $\delta(u,v)$ denote the minimal degree of a chain between u and v.

Note that δ is symmetric in u, v, that $\delta(u,v) = 0$ if and only if $u \leq v^*$, and that it is a non-decreasing fonction in the two variables.

Lemma 3.22 *For $u \in W_X$, we have $\delta(u, w_X) = \delta(u)$.*

Proof : If $u \in W/W_P$, let $x(u) = uP/P \in X = G/P$ denote the corresponding T-fixed point. By theorem 9.1 in [FW], $\delta(u, w_X) = \delta(w_X, u)$ is the minimal degree of a curve meeting $\{x(w_X)\}$ and $X(u^*)$. Since we can assume that this curve is T-invariant, it will pass through $x(w_X)$ and $x(v)$ with $v \in W/W_P$ and $v \leq u^*$. Applying the involution $x \mapsto w_0 x w_0 w_X$ of X, we deduce that this degree is the minimal degree of a curve through the base point and v^*. This minimal degree is $\delta(v^*)$, and is itself minimal when $v = u^*$, in which case it equals $\delta(u)$. $\square$

For $x \in X$, we now give a nice description of the subvariety $F_{d,x} := \{w \in F_d : x \in F_\omega\} \subset F_d$. Although this will not be used in the sequel, it shows that F_d generalizes very well F_1. In fact, the lines through a fixed point are parametrized by the closed P-orbit in the projectivization of the tangent space at this point, and we will show that more generally, the set of Y_ω's through a fixed point are parametrized by the closed P-orbit in the projectivization of the d-th normal space, according to the following definition :

Definition 3.23 Let $Z \subset \mathbb{P}V$ be a projective variety and let $z \in V - \{0\}$ such that $[z] \in Z$. Let d be an integer. We recall :

- The d-th affine tangent space $\widehat{T^d_{[z]}Z} \subset V$ is generated by the d-th derivatives at z of curves in the cone over Z.
- The d-th normal space $N^d_{[z]}Z$ is the quotient $\widehat{T^d_{[z]}Z}/\widehat{T^{d-1}_{[z]}Z}$.

Remark 3.24 (i) N^1 is the tangent space twisted by -1.

(ii) The d-th normal spaces to (co)minuscule homogeneous spaces are given in [LM, proposition 3.4]. If L denotes a Levi factor of P, they are irreducible L-modules.

For $\omega \in F_d$, let $Y_\omega \subset X$ denote the corresponding variety. For $x \in X$, let $F_{d,x} \subset F_d$ be the subvariety of ω's such that $x \in F_\omega$. If $Z \subset \mathbb{P}V$ is any subset, let $\langle Z \rangle \subset V$ denote the linear span of its cone in V. We have the following :

Proposition 3.25 *Let $x \in X$ and $\omega \in F_{d,x}$. Then $\dim(\langle Y_\omega\rangle/\widehat{T^{d-1}_x X}) = 1$. Moreover, the morphism*

$$\begin{array}{rcl} F_{d,x} & \to & \mathbb{P}N^d_x X \\ \omega & \mapsto & [\langle Y_\omega \rangle] \end{array}$$

is a closed immersion, with image the closed L-orbit in $\mathbb{P}N^d_x X$.

Proof : Let $x \in X$ be the base point, and let L be a Levi factor of P. By proposition 3.16, the subvarieties $F_{d,x} \subset F_d$ parametrize the elements ω in F_d which are incident to x in the sense of Tits geometries (namely the stabilizer of ω and that of x intersect along a parabolic subgroup), so $F_{d,x}$ is homogeneous under L. For example, if $X = \mathbb{G}(p, n)$, then $F_{d,x} \simeq \mathbb{G}(p-d, p) \times \mathbb{G}(d, n-p)$.

Therefore, to check that $\dim(\langle Y_\omega\rangle/\widehat{T^{d-1}_x X}) = 1$, it is enough to consider one particular example for ω; we leave this to the reader, as well as the fact that the corresponding class $[\langle Y_\omega\rangle] \in \mathbb{P}N^d_x X$ belongs to the closed L-orbit $\mathcal{O}$. For example, when $X = \mathbb{G}(p, n)$, if x denotes the linear space generated by $e_1, \ldots, e_p$ and ω the flag $(\langle e_1, \ldots, e_{p-d}\rangle, \langle e_1, \ldots, e_{p+d}\rangle)$, then any element in Y_ω is the linear span of $e_1, \ldots, e_{p-d}, f_1, \ldots, f_d$, with $f_i \in \langle e_1, \ldots, e_{p+d}\rangle$. Therefore, in the Plücker coordinates, this element is equivalent to a multiple of $e_1 \wedge \ldots \wedge e_d \wedge e_{p+1} \wedge \ldots \wedge e_{p+d}$ modulo $\widehat{T^{d-1}_x X}$.

We therefore have an L-equivariant morphism $F_{d,x} \to \mathcal{O}$, which must be an isomorphism for example because the two varieties have the same dimension. $\square$

3.6 Higher Gromov-Witten invariants

In this section, we deduce from the preceeding geometric results a way of computing higher degree Gromov-Witten invariants.

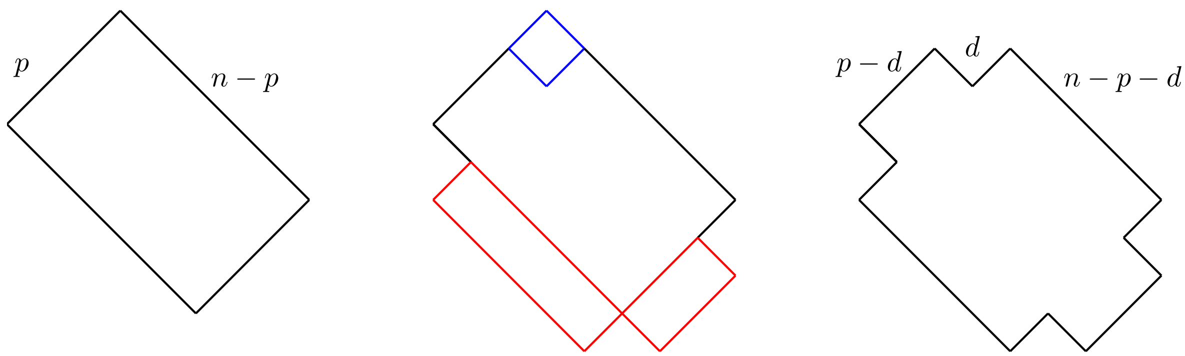
We keep the previous notations. As in the case of lines, the varieties Z_d and Y_d are (co)minuscule homogeneous variety. In particular they are endowed with well defined quivers Q_{Z_d} and Q_{Y_d}. The variety

Y_d is a Schubert subvariety of X and can thus be written as $X(w_{Y_d})$ for some $w_{Y_d} \in W_X$. We denote its Poincaré dual by $X(w^*_{Y_d})$ or Y^*_d.

We define as is the previous section the quivers Q_{I_d} and Q_{F_d} of I_d and F_d, by adding Q_{Z_d} at the end of Q_X (resp. by adding Q_{Z_d} at the end of $i_X(Q_{Y_d})$). These quivers correspond to the particular reduced decomposition obtained through the formula

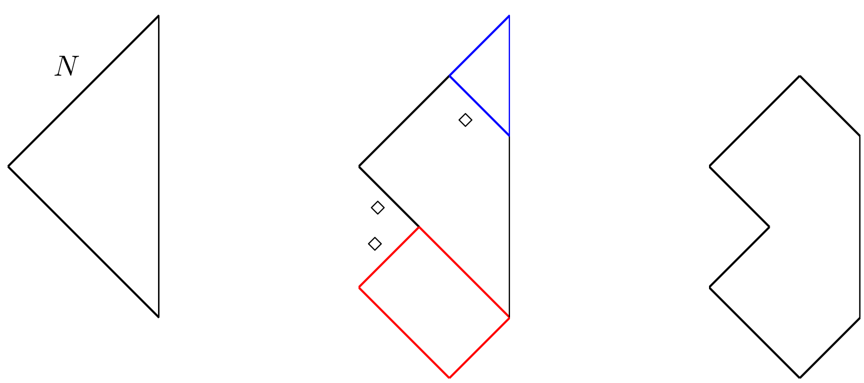
$$w_{I_d} = w_X w_{Z_d} = w_{Y_d} w_{F_d}.$$

Example 3.26 *Let $X = \mathbb{G}(p, n)$. We describe the quivers Q_X, Q_{Y_d}, Q_{Z_d} and Q_{F_d} and their different inclusions for $d \leq \min(p, n-p)$. We have already seen that Q_X is a $p \times (n-p)$ rectangle. We draw it as on the left picture.*



On the middle picture we have drawn Q_{I_d} and inside it, Q_{Z_d} in red and the complement of $i_X(Q_{w_{Y_d}})$ in Q_X in blue. The quiver on the right is Q_{F_d}.

The similar pictures in the isotropic cases are (we denote $\diamond = 2d, N = n-1$ in the quadratic case and $\diamond = d, N = n$ in the symplectic case):



Let $X(w)$ be a Schubert subvariety in X. Consider the Schubert subvariety $F_d(\widehat{w}) = q_d(p_d^{-1}(X(w))$ of F_d. The quiver of $F_d(\widehat{w})$ in Q_{F_d} is obtained as follows: attach Q_{Z_d} to the bottom end of $Q_w \cap i_X(Q_{w^*_{Y_d}})$. In particular, we have the inequality

$$\mathrm{codim}_{F_d}(F_d(\widehat{w})) \geq \mathrm{codim}_X(X(w)) - \dim(Y_d), \tag{6}$$

with equality if and only if $X(w) \subset X(w^*_{Y_d})$. (This generalizes inequalities obtained in [BKT] for (isotropic) Grassmannians.)

Now let $f : \mathbb{P}^1 \to X$ be a degree d morphism such that $f(\mathbb{P}^1)$ meets $X(w)$. Then there exists $y \in F_d$ such that $p_d(q_d^{-1}(y))$ meets $X(w)$, or equivalently, such that $y \in q(p^{-1}(X(w))) = F_d(\widehat{w})$. This is the key point to compute degree d Gromov-Witten invariants on X in terms of classical invariants on F_d.

Lemma 3.27 *Let $X(u)$, $X(v)$ and $X(w)$ be three Schubert subvarieties of X. Suppose that*

$$\mathit{codim}(X(u)) + \mathit{codim}(X(v)) + \mathit{codim}(X(w)) = \dim(X) + d \cdot c_1(X)$$

Then for g, g' and g'' three general elements in G, the intersection $g\cdot F(\widehat{u})\cap g'\cdot F(\widehat{v})\cap g''\cdot F(\widehat{w})$ is a finite set of reduced points. Moreover this finite set is empty if one of the Schubert varieties is not contained in $X(w^*_{Y_d})$.

Let y be a point in this intersection. Then the variety $p_d(q_d^{-1}(y))$ meets each of $g\cdot X(u)$, $g'\cdot X(v)$ and $g''\cdot X(w)$ in a unique point and these points are in general position in $p_d(q_d^{-1}(y))$.

Proof : Remark that the codimension condition, inequality (6) and equality (4) imply that

$$\begin{aligned}&\mathrm{codim}(F_d(\widehat{u}))+\mathrm{codim}(F_d(\widehat{v}))+\mathrm{codim}(F_d(\widehat{w}))\\ &\qquad\geq\ \mathrm{codim}(X_d(u))+\mathrm{codim}(X_d(v))+\mathrm{codim}(X_d(w))-3\dim(Y_d)\\ &\qquad=\ \dim(F_d),\end{aligned}$$

with equality if and only if the three Schubert varieties are contained in $X(w^*_{Y_d})$. Actually, this is true except for Grassmannians and for $d>\min(p,n-p)$, in which case the previous inequality is always strict. The first part of the lemma is thus implied by Bertini's theorem (see [Kl]).

Furthermore, by Bertini again, we may assume that any y in the intersection is general in $g\cdot F_d(\widehat{u})$, $g'\cdot F_d(\widehat{v})$ and $g''\cdot F_d(\widehat{w})$. In particular, by lemma 3.7 applied to $v=w_{Y_d}$, the variety $p_d(q_d^{-1}(y))$ meets each of $g\cdot X(u)$, $g'\cdot X(v)$ and $g''\cdot X(w)$ transversely in a unique point. Finally, the stabiliser of y acts transitively on $p_d(q_d^{-1}(y))$ and by modifying g, g' and g'' by elements in this stabiliser me may assume that the points are in general position in $p_d(q_d^{-1}(y))$. □

Corollary 3.28 *Let $X(u)$, $X(v)$ and $X(w)$ be three Schubert subvarieties of X. Suppose that the sum of their codimensions is $\dim(X)+d\cdot c_1(X)$. Then*

$$I_d([X(u)],[X(v)],[X(w)])=I_0([F_d(\widehat{u})],[F_d(\widehat{v})],[F_d(\widehat{w})]).$$

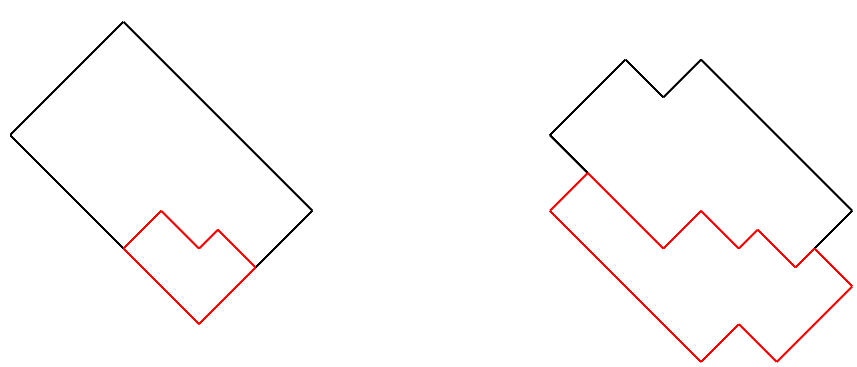
In particular, this invariant vanishes as soon as one of the three Schubert varieties is not contained in $X(w^*_{Y_d})$.

Proof : The image of any morphism f counting in the invariant $I_d([X(u)],[X(v)],[X(w)])$ is contained in a variety $p_d(q_d^{-1}(y))$ with $y\in g\cdot F_d(\widehat{u})\cap g'\cdot F_d(\widehat{v})\cap g''\cdot F_d(\widehat{w})$ for general elements g, g' and g'' in G. The preceding lemma implies that this intersection is either empty or a finite number of reduced points. Given such a y, the morphism f has to pass through three fixed general points in $p_d(q_d^{-1}(y))$, and by fact 3.18, there exists a unique such morphism.

In particular, for $X=\mathbb{G}(p,n)$ and $d>\min(p,n-p)$, all degree d Gromov-Witten invariants vanish. □

Remark 3.29 This result has been proved in [BKT] for (isotropic) Grassmannians through a case by case analysis. The vanishing condition generalizes the conditions of [Yo] for ordinary Grassmannians and of [BKT] for isotropic ones.

Example 3.30 *Let $X=\mathbb{G}(p,n)$. The quiver Q_w of a Schubert subvariety $X(w)$ has the following form (recall that it is the complement of the partition associated to w inside the rectangle $p\times(n-p)$):*



On the left picture we have drawn the quiver Q_X and inside it in red the Schubert subquiver Q_w. On the right we have the quiver Q_{F_d} and inside in red the subquiver of $F_d(\widehat{w})$.

4 The quantum Chevalley formula and a higher Poincaré duality

In this section we give, thanks again to the combinatorics of quivers, a simple combinatorial version of the quantum Chevalley formula (proposition 4.1). We also describe what we call a higher Poincaré duality (proposition 4.7): a duality on Schubert classes defined in terms of degree d Gromov-Witten invariants.

4.1 Quivers and the quantum Chevalley formula

A general quantum Chevalley formula has been obtained by W. Fulton and C. Woodward in [FW], following ideas of D. Peterson. In this subsection we recover this formula for any (co)minuscule homogeneous variety X, with a very simple combinatorial description in terms of quivers.

Indeed, since the codimension one Schubert subvariety H is certainly not contained in $X(w^*_{Y_d})$ for $d \geqslant 2$, the vanishing criterion of corollary 3.28 ensures that the quantum product with H only involves Gromov-Witten invariants degree zero and one.

Let us first give a few more notations to describe our quantum Chevalley formula. If i is a peak of Q_w, we denote by $Q_{w(i)}$ the full subquiver of Q_w obtained by removing the vertex i and by $w(i)$ the corresponding element in W. Embed the quivers Q_X, Q_F and Q_Z in Q_I as explained in subsection 3.3. If $X(w)$ is a Schubert subvariety of X, consider the Schubert variety $F(i_F(\widehat{w}))$ in F. If it exists, denote by w_q the element in W such that $F(i_F(\widehat{w})) = F(\widehat{w}_q)$.

Proposition 4.1 *For any Schubert subvariety $X(w)$ of X, we have*

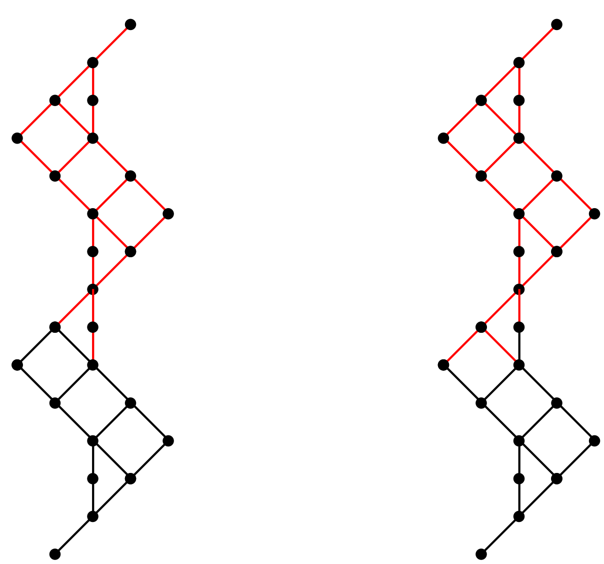
$$[X(w)] * [H] = \sum_{i \in p(Q_w)} [X(w(i))] + q[X(i_X(w_q))].$$

Proof : The degree zero part of the right hand side is the classical product $[X(w)] \cdot [H]$, for which we have just reformulated the classical Chevalley formula in terms of quivers.

For the degree one part, we need to compute the Gromov-Witten invariants $I_1([H], [X(w)], [X(v)])$ for all $v \in W_X$. Since $q(p^{-1}(H)) = F$, this amounts by corollary 3.11 to compute $I_0([F], [F(\widehat{w})], F(\widehat{v})])$. By Poincaré duality on F this invariant is zero unless $\widehat{v} = i_F(\widehat{w})$, in which case it is equal to one. But this exactly means that w_q exists and is equal to v. $\square$

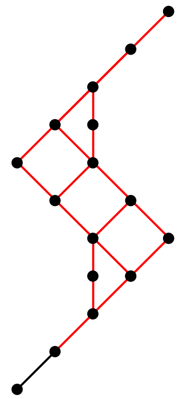
Definition 4.2 We will say that $X(w)$ and $X(w_q)$ are *1-Poincaré dual.*

Example 4.3 *Let X, $X(w)$ and $X(v)$ as in example 2.13 (ıı). Then the quivers of $F(i_F(\widehat{w}))$ and $F(i_F(\widehat{v}))$ are the following:*



$F(i_F(\widehat{w}))$ $\qquad$ $F(i_F(\widehat{v}))$

and in particular $F(i_F(\widehat{w}))$ cannot be of the form $F(\widehat{u})$ for some u. On the contrary for u with the following quiver,



we have $F(i_F(\widehat{v})) = F(\widehat{u})$. In particular we obtain with the notations of subsection 2.3.1:

$$H * \sigma'_{12} = \sigma_{13} \qquad \textit{and} \qquad H * \sigma''_{12} = \sigma_{13} + q\sigma_1.$$

4.2 Higher quantum Poincaré duality

Poincaré duality can be reformutated as follows: there exists an involution of W_X, given by $v \mapsto v^* = w_0 v w_0 w_X$, such that

$$I_0([X], [X(v)], [X(w)]) = \delta_{w,v^*}.$$

We have seen that for degree one invariants, the hyperplane class $[H]$ plays the role of $[X]$: $X(v)$ and $X(w)$ are 1-Poincaré dual (see definition 4.2) if and only if

$$I_1([H], [X(v)], [X(w)]) = 1.$$

More generally, the class $[Y_d^*]$ will play the role of $[X]$ for degree d Gromov-Witten invariants. We will define an involution $v \mapsto v_{q^d}$ of a subset of W_X, with a simple combinatorial interpretation, and such that

$$I_d([Y_d^*], [X(v)], [X(w)]) = \delta_{w,v_{q^d}}$$

(with the understanding that if v_{q^d} is not defined, then the invariant is zero).

Before giving a precise definition of v_{q^d}, let us describe Poincaré duality on F_d. As for $X = F_0$ or $F = F_1$, compiling reduced expressions $w_X = s_{\beta_1} \cdots s_{\beta_N}$ and $w_{Z_d} = s_{\beta'_1} \cdots s_{\beta'_{M_d}}$, where $M_d = \dim(Z_d)$, we obtain the reduced expression

$$w_{F_d} = s_{\beta_2} \cdots s_{\beta_N} s_{\beta'_1} \cdots s_{\beta'_{M_d}}.$$

Modulo commutation relations, this expression is symmetric, that is, of the form $s_{\gamma_1} \cdots s_{\gamma_{R_d}}$ with $R_d = \dim(F_d)$ and $i(\gamma_k) = \gamma_{R_d+1-k}$. The associated quiver $Q_{w_{F_d}}$ is symmetric and we denote by i_{F_d} the induced involution on subquivers. The same proof as for proposition 2.12 gives the following result:

Proposition 4.4 *Let $F_d(w)$ be a Schubert subvariety of F_d such that $w = s_{\gamma_{k+1}} \cdots s_{\gamma_{R_d}}$. Then the classes $[F_d(w)]$ and $[F_d(i_{F_d}(w))]$ are Poincaré dual.*

Remark 4.5 (i) Beware that not all Schubert varieties $F_d(w), w \in W_{F_d}$ satisfy the hypothesis of the proposition. This is because F_d is not minuscule and in consequence there may be braid relations. However, all Schubert varieties $F_d(w)$ associated to a Schubert subquiver $Q_{F_d}(w)$ of Q_{F_d} satisfy the property.

(ii) We will denote by $F_d(u^\star)$ the Poincaré dual of $F_d(u)$.

The quiver Q_{F_d} contains $Q_{w^*_{Y_d}} = Q_{Y_d^*}$ and is symmetric. We denote by i_{F_d} the associated involution. The subquiver

$$Q_{w_{T_d}} := Q_{Y_d^*} \cap i_{F_d}(Q_{Y_d^*}) \subset Q_{Y_d^*} \subset Q_X$$

is a symmetric Schubert subquiver. We let $T_d = X(w_{T_d})$ and denote by i_{T_d} the involution on Q_{T_d}

The varieties T_d are given by the following table. Observe that they are always smooth, and that the vertices of $Q_{T_{d-1}}$ are those under the vertex $(\iota(\alpha), d)$, where ι is the Weyl involution of the simple roots and α is the root defining X.

X	d	T_d
$\mathbb{G}(p,n)$	$d \leq \min(p, n-p)$	$\mathbb{G}(p-d, n-2d)$
$\mathbb{G}_\omega(n,2n)$	$d \leq n$	$\mathbb{G}_\omega(n-d, 2n-2d)$
$\mathbb{G}_Q(n,2n)$	$d \leq \frac{n}{2}$	$\mathbb{G}_Q(n-2d, 2n-4d)$
$\mathbb{Q}^n$	$d=1$	$\mathbb{P}^1$
	$d=2$	$\{\mathrm{pt}\}$
E_6/P_1	$d=1$	$\mathbb{P}^5$
	$d=2$	$\{\mathrm{pt}\}$
E_7/P_7	$d=1$	$\mathbb{Q}^{10}$
	$d=2$	$\mathbb{P}^1$
	$d=3$	$\{\mathrm{pt}\}$

Definition 4.6 The application $v \mapsto v_{q^d}$ is defined for all $v \in W_X$ such that Q_v is contained in $Q_{w_{T_d}}$, or equivalently $X(v) \subset T_d$, by

$$Q_{v_{q^d}} = i_{F_d}(Q_v) \cap Q_X = i_{T_d}(Q_v \cap Q_{T_d}).$$

Otherwise said, the map $v \mapsto v_{q^d}$ is given by Poincaré duality inside T_d.

Proposition 4.7 *The Gromov-Witten invariant $I_d([X(w^*_{Y_d})], [X(v)], [X(w)])$ vanishes unless $w = v_{q^d}$. In that case the invariant is equal to one.*

Proof : The proof is similar to that of proposition 4.1. From corollary 3.28, we know that

$$I_d([X(w^*_{Y_d})], [X(v)], [X(w)]) = I_0([F_d(\widehat{w^*_{Y_d}})], [F_d(\widehat{v})], [F_d(\widehat{w})]).$$

But $F_d(\widehat{w^*_{Y_d}}) = F_d$, so this invariant is trivial unless $[F_d(\widehat{v})]$ and $[F_d(\widehat{w})]$ are Poincaré dual in F_d. But proposition 4.4 applies to $F_d(\widehat{v})$ and the invariant vanishes unless the quivers of $F_d(\widehat{v})$ and $F_d(\widehat{w})$ are symetric under i_{F_d} and in that case the invariant equals one. This is equivalent to $w = v_{q^d}$. $\square$

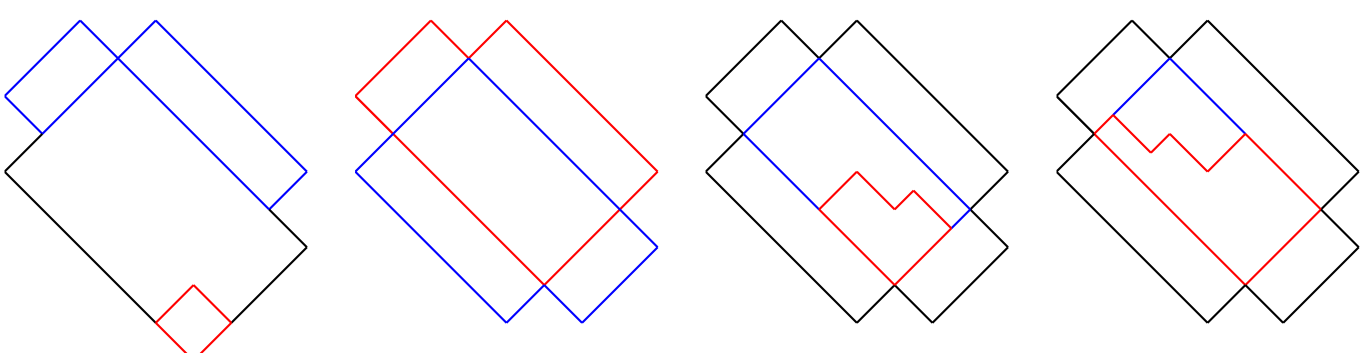
Before dealing with examples, let us state the following lemma generalizing Lemma 3.13 for $d \geq 2$; the proof is the same. It will be useful together with Lemma 3.12 to prove the vanishing of some Gromov-Witten invariants.

Lemma 4.8 *Let u and v in W_{F_d} such that $F_d(u)$ and $F_d(v)$ are represented by Schubert subquivers $Q_{F_d}(u)$ and $Q_{F_d}(v)$ of Q_{F_d}.*

($\imath$) If $Q_{F_d}(u) \subset Q_{F_d}(v)$ then $F_d(u) \subset F_d(v)$ (see Fact 2.11 ($\imath$)).

($\imath\imath$) Conversely, if $F_d(u) \subset F_d(v)$, then we have the inclusion $Q_{F_d}(u) \cap i_{F_d}(Q_{Z_d}) \subset Q_{F_d}(v) \cap i_{F_d}(Q_{Z_d})$.

Example 4.9 *($\imath$) Suppose again that $X = \mathbb{G}(p,n)$ is a Grassmannian. We give on the left picture the quiver Q_{F_d} with inside it in blue the quiver $i_{F_d}(Q_{Z_d})$ and in red the quiver Q_{Y_d} one can add to $i_{F_d}(Q_{Z_d})$ to get Q_X.*

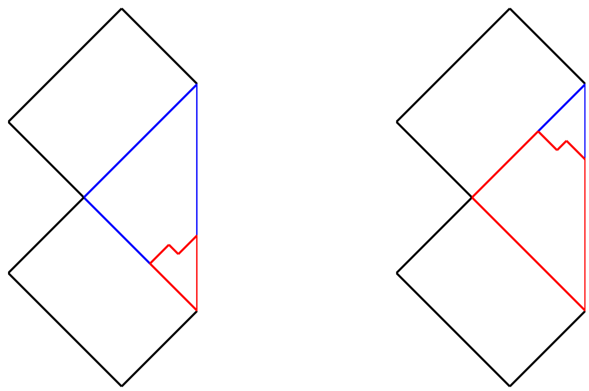


In the second picture we have drawn inside Q_{F_d} the quiver $Q_{Y_d^*}$ in red and the quiver $i_{F_d}(Q_{Y_d^*})$ in blue. In the third one, we have Q_{T_d} in blue and inside it Q_w in red. Finally on the right picture we have the quiver $Q_{w_{q^d}}$ in red.

In terms of partitions, consider a Schubert class $[X(\lambda)]$, where λ is a partition whose Ferrers diagram is contained in the rectangle $p \times (n-p)$. For $d \leqslant \min(p, n-p)$, this class admits a d-Poincaré dual if and only if $\lambda_d = n-p$ and $\lambda_p \geqslant d$. Then λ is uniquely defined by the partition μ, whose diagram is contained in the rectangle $(p-d) \times (n-p-d)$ (the quiver Q_{T_d}), such that $\mu_i = \lambda_{d+i} - d$. Let μ^* be the partition complementary to μ in that smaller rectangle. Then the d-Poincaré dual $[X(\lambda)]$ is the class $[X(\lambda_{q^d})]$, where the partition λ_{q^d} is defined by

$$\lambda_{q^d,d} = n-p, \qquad \lambda_{q^d,i} = d + \mu^*_{i-d} \quad \text{for } i > d.$$

(ii) For an isotropic Grassmannian $\mathbb{G}_\omega(n, 2n)$ (resp. $\mathbb{G}_Q(n, 2n)$), the picture similar to that for ordinary Grassmannians is:



Recall that the Schubert classes are indexed by strict partitions λ made of integers smaller or equal to N (recall the notation of example 3.26)– this is often denoted $\lambda \subset \rho_N$, where $\rho_N = (N, N-1, \dots, 2, 1)$.

The Poincaré dual of the Schubert class $[X(\lambda)]$ is the class $[X(\lambda^*)]$, where λ^* is the partition whose parts complement the parts of λ in the set $\{1, \dots, N\}$. More generally, the Schubert class $[X(\lambda)]$ has a d-Poincaré dual $[X(\lambda_{q^d})]$ if and only if $\lambda_\diamond = N - \diamond$; in this case, denote by $\mu \subset \rho_{N-\diamond}$ the partition defined by $\mu_i = \lambda_{N-\diamond+i}$. Then we have

$$(\lambda_{q^d})_i = \begin{cases} N-i & \text{if} \quad i \leq \diamond \\ \mu^*_{i-\diamond} & \text{if} \quad i > \diamond, \end{cases}$$

where μ^* is the complement of μ inside $\rho_{N-\diamond}$.

(iii) If $X = E_6/P_1$ and $d = 2$, then $[Y_d] = \sigma_8$ and the only Schubert class $[X(w)]$ such that $\sigma_8 * [X(w)]$ has a non trivial degree two term is $[X(w)] = \sigma_{16}$. In this case we have

$$\sigma_8 * \sigma_{16} = q^2 \sigma_0.$$

Indeed, the q^2 term comes from Poincaré duality, we proved in example 3.14 that all degree one invariants $I_1(\sigma_8, \sigma_{16}, \sigma_u)$ vanish and for dimension reasons, there is no q^0 term.

The previous observation can be generalized as follows:

Proposition 4.10 *Let $d_{\max}$ be the maximal power of q in the quantum product of two Schubert classes. Then we have the following formulae:*

$$\begin{aligned} [\{\mathrm{pt}\}] * [\{\mathrm{pt}\}] &= q^{d_{\max}} [Y_{d_{\max}}], \\ [Y^*_{d_{\max}}] * [\{\mathrm{pt}\}] &= q^{d_{\max}} [X]. \end{aligned}$$

The values of $d_{\max}$ are the following. For a Grasmannian $\mathbb{G}(p, n)$, $d_{\max} = \min(p, n-p)$. For $\mathbb{G}_Q(n, 2n)$, $d_{\max} = [n/2]$ and for $\mathbb{G}_\omega(n, 2n)$, $d_{\max} = n$. Finally, $d_{\max} = 2$ for quadrics or the Cayley plane, while $d_{\max} = 3$ for the Freudenthal variety.

Proof : If $q^d[X(w)]$ appears in the product $[\{\mathrm{pt}\}] * [\{\mathrm{pt}\}]$, then $X(w)$ must contain Y_d by corollary 3.28. In particular we must have

$$\dim(X) + d \cdot c_1(X) = 2\dim(X) + \dim(X(w)) \geq 2\dim(X) + \dim(Y_d).$$

But since the d-Poincaré dual to the class of a point is our variety T_d, we have the relation

$$\dim(X) + d \cdot c_1(X) = \dim(X) + \mathrm{codim}(T_d) + \dim(Y_d).$$

Comparing with the previous inequality, we get $\mathrm{codim}(T_d) \geqslant \dim(X)$, hence $T_d = X$ and $d = d_{\max}$, and $\dim(X(w)) = \dim(Y_d)$, thus $X(w) = Y_d$. In particular we only have the term $q^{d_{\max}}[Y^*_{d_{\max}}]$ in $[\{\mathrm{pt}\}] * [\{\mathrm{pt}\}]$. The higher Poincaré duality implies that the coefficient is one and the first identity follows.

To prove the second one, using the first identity and the associativity of the quantum product, we see that we only need to prove that if q^d appears in the product $[Y^*_{d_{\max}}] * [\{\mathrm{pt}\}]$, then $d = d_{\max}$. Let us set $v = w^*_{Y_{d_{\max}}}$ and $w = 1$ so that $X(w) = \{\mathrm{pt}\}$. The Schubert variety $F_d(\widehat{w})$ contains $F_d(\widehat{v}^\star)$ if and only if there is an inclusion $Q_{F_d}(\widehat{v}^\star) \cap i_{F_d}(Q_{Z_d}) \subset Q_{F_d}(\widehat{w}) \cap i_{F_d}(Q_{Z_d})$ of quivers (cf Lemma 4.8). But both quivers $Q_{F_d}(\widehat{v}^\star)$ and $Q_{F_d}(\widehat{w})$ are contained in $i_{F_d}(Q_{Z_d})$ and if $d < d_{\max}$, the quiver of $F_d(\widehat{w})$ does not contain the quiver of $F_d(\widehat{v}^\star)$. The Lemma 3.12 gives the vanishing of $I_0([Y^*_{d_{\max}}], [\{\mathrm{pt}\}], [F_d(\widehat{u})])$ for any $u \in W_X$ and in particular q^d appears in $[Y^*_{d_{\max}}] * [\{\mathrm{pt}\}]$ if and only if $d = d_{\max}$. □

4.3 The smallest power in a quantum product of Schubert classes

In [FW], Fulton and Woodward described the minimal power of q that can appear in the quantum product $[X(u)] * [X(v)]$ of two Schubert classes. In this subsection we give a new combinatorial description of this minimal power for (co)minuscule homogeneous varieties. This generalizes the reinterpretation of Fulton and Woodward's result by A. Buch in [Bu, Theorem 3].

Let $u \in W_X$, we define the element $\overline{u} \in W_X$ by $Q_{\overline{u}} = i_X(i_{T_d}(Q_u \cap Q_{T_d}))$.

Proposition 4.11 *Let $X(u)$ and $X(v)$ be two Schubert subvarieties in X. If q^d appears in $[X(u)] * [X(v)]$ then the quiver $X(\overline{u}^*)$ is a subquiver of $X(v)$.*

Proof : The hypothesis that q^d appears in the quantum product $[X(u)]*[X(v)]$ is equivalent to the existence of an element $w \in W_X$ such that $I_d([X(u)], [X(v)], [X(w)]) \neq 0$. By corollary 3.28, this is equivalent to the non vanishing of the classical invariant $I_0([F_d(\widehat{u})], [F_d(\widehat{v})], [F_d(\widehat{w})])$. This in particular implies that the product $[F_d(\widehat{u})] \cdot [F_d(\widehat{v})]$ is non zero and, by Lemma 3.12, that $F_d(\widehat{v}) \supset F_d(\widehat{u}^\star)$. Thanks to Lemma 4.8, we have the inclusions $Q_{F_d}(\widehat{u}^\star) \cap i_{F_d}(Q_{Z_d}) \subset Q_{F_d}(\widehat{v}) \cap i_{F_d}(Q_{Z_d})$ of quivers. Because $Q_{F_d}(\widehat{u}^\star)$ is contained in $Q_{F_d}(Z_d)$ and $Q_{F_d}(\widehat{v}) \cap i_{F_d}(Q_{Z_d})$ contains Q_{Z_d}, this inclusion is equivalent to $Q_{F_d}(\widehat{u}^\star) \cap Q_{T_d}) \subset Q_{F_d}(\widehat{v}) \cap Q_{T_d}$. This is equivalent to the inclusion $i_X(Q_{\overline{u}}) \subset Q_v$ or $X(\overline{u}^*) \subset X(v)$. □

The following corollary gives the smallest power of q in a quantum product. It is a generalization of lemma 3.12 giving a condition for q^0 to appear in that product. It is also a generalisation of theorem 3 in [Bu]:

Corollary 4.12 *Let $X(u)$ and $X(v)$ be two Schubert subvarieties in X. The smallest power q^d that appears in $[X(u)] * [X(v)]$ is the smallest d such that*

$$Q_u \subset Q_{w^*_{Y_d}}, \qquad Q_v \subset Q_{w^*_{Y_d}}, \qquad i_X(Q_{\overline{u}}) \subset Q_v.$$

Proof : We already know by the vanishing criterion of corollary 3.28 and the previous proposition that the conditions are necessary for the product $[X(u)] * [X(v)]$ to have a q^d term.

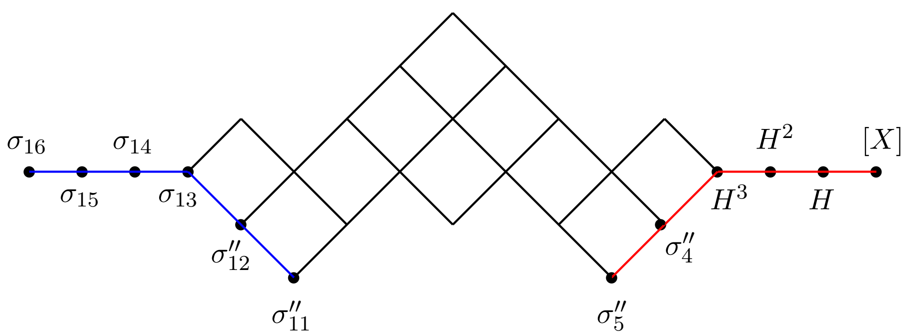
Conversely, let us denote by $\widetilde{u}$ the element in W_X such that $Q_{\widetilde{u}} = Q_u \cap Q_{T_d}$. Since $Q_{\widetilde{u}}$ is contained in Q_u, the product $H^{*a} * \sigma_u$ with $a = \dim X(u) - \dim X(\widetilde{u})$ contains $\sigma_{\widetilde{u}} = \sigma_{w_{T_d}} \cdot \sigma_{\overline{u}}$. Multiplying by $\sigma_{w^*_{Y_d}}$ gives by higher Poincaré duality that the product $\sigma_{w^*_{Y_d}} * H^{*a} * \sigma_u$ contains $\sigma_{w^*_{Y_d}} * \sigma_{w_{T_d}} \cdot \sigma_{\overline{u}} = q^d \sigma_{\overline{u}}$. Finally, the product $\sigma_{w^*_{Y_d}} * H^{*a} * \sigma_u * \sigma_v$ contains $q^d \sigma_{\overline{u}} * \sigma_v$ which has a non zero q^d term by Lemma 3.12 and the hypothesis $i_X(Q_{\overline{u}}) \subset Q_v$. By non negativity of the invariants, the product $\sigma_u * \sigma_v$ must contain a power in q smaller or equal to d. □

5 The quantum cohomology of the exceptional minuscule varieties

In this section we apply our quantum Chevalley formula to the exceptional minuscule varieties. In turns out that, together with our computation in example 3.14, this suffices to deduce the whole quantum Chow ring from the classical one.

5.1 The quantum Chow ring of the Cayley plane

The quantum Chevalley formula for the Cayley plane $\mathbb{OP}^2$ can be conveniently visualized on the Hasse diagram. The following follows from subsection 4.2 and example 3.3. The Schubert classes having a 1-Poincaré dual class are those in the Bruhat interval $[\sigma_{16}, \sigma''_{11}]$, represented in blue on the picture below. (Note that σ''_{11} is the class of the Schubert variety T_1, with the notations of the previous subsection.) For such a class σ, the 1-Poincaré dual is obtained by first applying the obvious symmetry in this diagram, and then the usual Poincaré duality. This means that the q-term in $\sigma * H$ is the Schubert class corresponding to σ in the isomorphic interval $[\sigma''_5, [X]]$, in red on the picture.



Here is the quantum version of Proposition 2.2.

Theorem 5.1 *Let $\mathcal{H}_q := \mathbb{Z}[h, s, q]/(3hs^2 - 6h^5 s + 2h^9, s^3 - 12h^8 s + 5h^{12} - q)$. Mapping h to H, s to σ'_4 and preserving q yields an isomorphism of graded algebras*

$$\mathcal{H}_q \simeq QA^*(\mathbb{OP}^2).$$

Proof : By proposition 2.2 and [FP, proposition 11], it is enough to show that the displayed relations hold in $QA^*(\mathbb{OP}^2)$. Recall that the index of the Cayley plane is twelve. The relation $3H * \sigma^{*2} - 6H^{*5} * \sigma + 2H^{*9} = 0$ holds because its degree is strictly less than 12 and it holds in the classical Chow ring.

The Fano variety of lines through a given point can be identified with the spinor variety $G_Q(5, 10)$ in its minimal embedding (the projectivization of a half-spin representation). In particular its degree equals 12 (see [LM, 3.1]). Applying proposition 3.1, we get the relation

$$H^{*12} = H^{12} + 12q, \tag{7}$$

which we could also deduce from the quantum Chevalley formula. Now we use the result of example 3.14, according to which the multiplication of σ_8 by any class of degree four does not require any quantum correction. Since $\sigma_8 = \sigma^2 + 2H^4\sigma - H^8$, we get:

$$\begin{array}{rcccl} \sigma^{*3} + 2H^{*4} * \sigma^{*2} - H^{*8} * \sigma & = & \sigma'_{12} & = & 15H^8\sigma - 19/3H^{*12} + 76q, \\ H^{*4} * \sigma^{*2} + 2H^{*8} * \sigma - H^{*12} & = & \sigma'_{12} + \sigma''_{12} & = & 4H^8\sigma - 5/3H^{*12} + 20q. \end{array}$$

Indeed, for the first line we have used that $\sigma_8 * \sigma'_4 = \sigma_8 \cdot \sigma'_4 = \sigma'_{12}$, the expression for σ'_{12} obtained in the proof of Proposition 2.2, and the identity (7). For the second line we have used that $\sigma_8 H^4 = \sigma'_{12} + \sigma''_{12}$

Eliminating $H^8\sigma$, we get $4\sigma^{*3} - 7H^{*4} * \sigma^{*2} - 34H^{*8} * \sigma + 46/3H^{*12} - 4q = 0$. Taking into account the relation already proved, this yields $4\sigma^{*3} - 48H^{*8} * \sigma + 20H^{*12} - 4q = 0$, as claimed. $\square$

We can be more specific about the quantum multiplication of Schubert classes. The first quantum corrections appear in degree twelve. In this degree, the only cases which do not follow directly from the quantum Chevalley formula are the products of a degree eight class by a degree four class. We have

$$\sigma_8' * \sigma_4'' = \sigma_8'\sigma_4'' + q, \qquad \sigma_8'' * \sigma_4' = \sigma_8''\sigma_4' + q,$$

while the other products have no quantum correction.

In fact, to prove this, let us first compare the quantum monomials $H^{*i} * \sigma^{*j}$ of degree 12 with the corresponding classical products. The classical Chevalley formula gives

$$H^3 \cdot \sigma^2 = 2\sigma_{11}'' + 6\sigma_{11}' \text{ and } H^7 \cdot \sigma = 5\sigma_{11}'' + 14\sigma_{11}'.$$

Therefore, our quantum Chevalley formula yields

$$H^4 * \sigma^2 = H^4 \cdot \sigma^2 + 2q \text{ and } H^8 * \sigma = H^8 \cdot \sigma + 5q.$$

Since $\sigma \cdot \sigma_8 = \sigma * \sigma_8$, it follows that $\sigma^{*3} = \sigma^3 + q$. The claims about quantum products of Schubert classes of degree 4 and 8 follow directly.

It is then easy, inductively, to obtain the following formulae:

$$\begin{aligned}
\sigma_{12}' &= \sigma^3 + 2H^4\sigma^2 - H^8\sigma, \\
\sigma_{12}'' &= -\sigma^3 - H^4\sigma^2 + 3H^8\sigma - H^{12}, \\
\sigma_{13} &= H\sigma^3 + 2H^5\sigma^2 - H^9\sigma, \\
\sigma_{14} &= 2H^6\sigma^2 + 11H^{10}\sigma - 5H^{14}, \\
\sigma_{15} &= -H^3\sigma^3 + 2H^7\sigma^2 + 23H^{11}\sigma - 10H^{15}, \\
\sigma_{16} &= \sigma^4 - 2H^4\sigma^3 - 10H^8\sigma^2 + H^{16}.
\end{aligned}$$

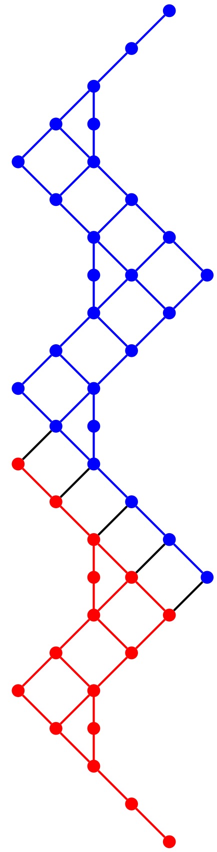
We have deliberately omitted the signs for the quantum product. In fact we have:

Proposition 5.2 *The previous Giambelli type formulas for the Schubert classes of the Cayley plane hold in the classical as well as in the quantum Chow ring.*

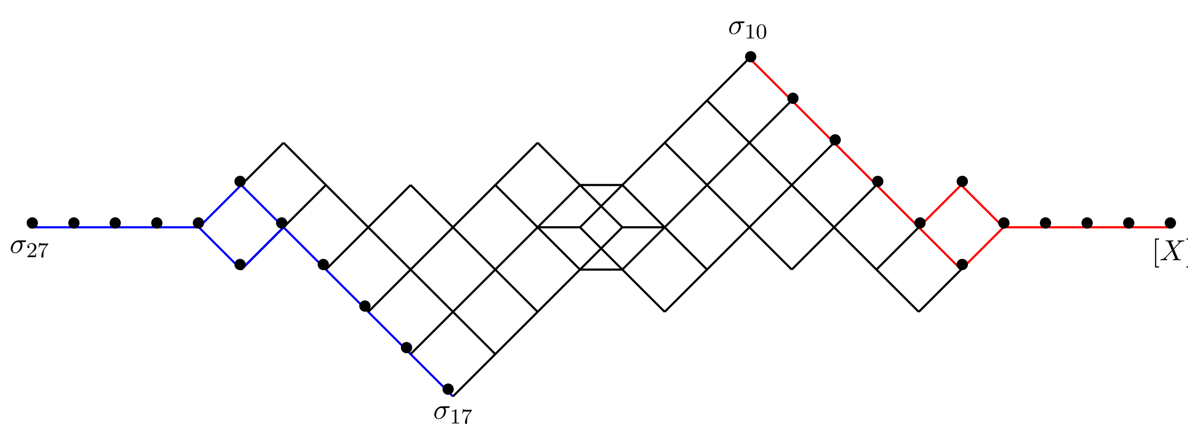
Of course in degree smaller than twelve, this is also the case of the Giambelli type formulas given in the proof of Proposition 2.2. The coincidence of classical and quantum Giambelli formulas already appeared for minuscule homogeneous varieties of classical type (see [Be, KT2]).

5.2 The quantum Chow ring of the Freudenthal variety

The quantum Chevalley formula for the Freudenthal variety E_7/P_7 can again easily be visualized on the Hasse diagram. The quiver of the Fano variety of lines in E_7/P_7 looks as follows (the quiver of E_7/P_7 is in blue and the quiver of Z is red):



The Schubert classes having a 1-Poincaré dual class are those in the Bruhat interval $[\sigma_{27}, \sigma_{17}]$, represented in blue on the picture below, σ_{17} being the class of T_1. For such a class σ, the 1-Poincaré dual is obtained by first applying the obvious symmetry of this interval, and then the usual Poincaré duality. So the q-term in $\sigma * H$ is the Schubert class corresponding to σ in the isomorphic interval $[\sigma_{10}, [X]]$, in red on the picture.



As for the Cayley plane, we can deduce all the quantum products of Schubert classes in degree 18. For example, let c, d be classes of degrees 5 and 13. Since the product by the hyperplane class defines an isomorphism between A^{12} and A^{13}, we can write $d = He$ for some class e of degree 12, either in the classical or the quantum Chow ring. Using the associativity of the quantum product, we get

$$c * d = c * (He) = c * H * e = (c * e) * H = (ce) * H.$$

This can be computed from the classical intersection product and the quantum Chevalley formula.

A priori, this method does not work if we want to compute the quantum product of two classes of degree 9. Indeed the product with the hyperplane class does not define an isomorphism between A^8 and A^9. Nevertheless, the fact that the quantum correction of a product like $\sigma_9 * \sigma_9'$ is a *non-negative* integer

multiple of q allows us to carry the computation over. For example, we have

$$(\sigma_9' + \sigma_9'') * \sigma_9 = H\sigma_8'' * \sigma_9 = (\sigma_8''\sigma_9) * H = (2\sigma_{17}' + 5\sigma_{17}'') * H.$$

Since the class σ_{17} does not appear, the quantum Chevalley formula shows that there is no quantum correction in this product, and therefore there is none either in the two products $\sigma_9' * \sigma_9$ and $\sigma_9'' * \sigma_9$.

Our conclusion is the following:

Proposition 5.3 *Among the quantum products of two Schubert classes of total degree* 18*, the only ones that require a quantum correction are*

$$\begin{array}{rcl} \sigma_{17} * H & = & \sigma_{17}H + q, \\ \sigma_{13}' * \sigma_5' & = & \sigma_{13}'\sigma_5' + q, \\ \sigma_{13}'' * \sigma_5'' & = & \sigma_{13}''\sigma_5'' + q, \\ \sigma_9 * \sigma_9 & = & (\sigma_9)^2 + q, \\ \sigma_9' * \sigma_9' & = & (\sigma_9')^2 + q, \\ \sigma_9'' * \sigma_9'' & = & (\sigma_9'')^2 + q. \end{array}$$

This completely determines the quantum Chow ring. The quantum version of Theorem 2.5 is the following.

Theorem 5.4 *Let* $\mathcal{H} = \mathbb{Z}[h, s, t, q]/(s^2 - 10sh^5 + 2th + 4h^{10}, 2st - 12sh^9 + 2th^5 + 5h^{14}, t^2 + 922sh^{13} - 198th^9 - 385h^{18} - q)$*. Mapping h to H, s to σ_5', t to σ_9 and preserving q yields an isomorphism of graded algebras*

$$\mathcal{H}_q \simeq QA^*(E_7/P_7).$$

Proof : As for the proof of Theorem 5.1, we just need to prove that the displayed relations hold in $QA^*(E_7/P_7)$. This is clear for the first two, since they are of degree smaller that 18 and they hold in the classical Chow ring by Theorem 2.5. That the third equation also holds is a direct consequence of Proposition 5.3. $\square$

As we did for the Cayley plane, we could also derive Giambelli type formulas for the Schubert classes, holding both in the classical and the quantum Chow ring.

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